EMBEDDING PLANAR COMPACTA IN PLANAR CONTINUA WITH APPLICATION: HOMOTOPIC MAPS BETWEEN PLANAR PEANO CONTINUA ARE CHARACTERIZED BY THE FUNDAMENTAL GROUP HOMOMORPHISM.

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Abstract. The CAT(0) geometry of a planar PL disk (determined by internal paths of minimal length) is employed to prove every planar compactum with connected complement can be an embedded in a cellular planar continuum by attaching a null sequence of arcs with disjoint interiors.

This leads to a proof that two based maps from a planar Peano continuum to a planar set are homotopic iff they induce the same homomorphism between fundamental groups.

1. Introduction

The capacity to recognize homotopic maps plays a central role in classifying planar continua up to homotopy equivalence.

The second of two main results (Theorems 9 and 10) establishes if $X \subset R^2$ is a locally path connected (i.e. Peano) continuum, and $Y \subset R^2$ is arbitrary, and if $f, g : (X, p) \to (Y, q)$ are based maps, then f is homotopic to g if and only if the induced maps $f_*, g_* : \pi_1(X, p) \to \pi_1(Y, q)$ satisfy $f_* = g_*$.

The above statement is generally false for planar continua, due for example to the existence of cellular noncontractible planar continua. Such examples also confirm the potential failure of the conclusion of Whitehead's Theorem [10] [11] (maps between CW complexes which induce isomorphisms on homotopy groups are homotopy equivalences)

For planar Peano continua X and Y, it is an open question whether $f : X \to Y$ is a homotopy equivalence precisely if f_* induces an isomorphism between fundamental groups.

In the simplest nontrivial case, (the Hawaiian earring HE, the union of a null sequence of circles joined at a common point), the group $\pi_1(HE)$ is uncountable and not free [5], it naturally injects into the inverse limit of finite free groups, and its elements can be understood as "infinite words" in generators $x_1, x_2, ...$ such that each letter appears finitely many times [9]. Remarkably, Eda [6] proved all homomorphisms of the Hawaiian earring group are (up to a change of base point isomorphism) induced by maps, and hence the self homotopy equivalences $f : HE \to HE$ are precisely the maps such that f induces an isomorphism of $\pi_1(HE)$.

More generally, positive answers are emerging in case X is 1 dimensional [7] or homotopy equivalent to a 1 dimensional planar Peano continuum [4].

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Most generally (e.g. [12]) , Theorem 10 of the paper at hand immediately yields a partial answer**:** If $X, Y \subset R^2$ are Peano continua, then a map $f : X \to Y$ is a homotopy equivalence if f_* is an isomorphism and f_*^{-1} is induced by a map (Corollary 2).

To prove Theorem 10, we must confront the fact that if $A \subset int(D^2)$ is an **arbitrary** compactum in the closed unit disk D^2 such that $\dim(A) \leq 1$, and if $\{U_n\} \subset int(D^2)\backslash A$ is a null sequence of disjoint round open disks converging (limit supremum) to A, then $X = D^2\backslash\{U_1 \cup U_2...\}$ is a Peano continuum such that $X\backslash int(X) = Z = A \cup \partial U_1 \cup \partial U_2...$

Since A is arbitrary, critical to the proof of Theorem 10, is our other main result (Theorem 9), which establishes that every planar compactum $Z \subset R^2$ (such that $R^2\backslash Z$ is connected) can be embedded in a nonseparating planar continuum $W \subset R^2$ by attaching a null sequence of topological arcs (with disjoint interiors) to Z.

Theorem 9 effectively generalizes some recent work of Blokh, Misiurewicz, and Oversteegen [1], who employ a similar strategy in the special case the nontrivial components of Z form a null sequence of Peano continua. In [1], the resulting continuum W is a Peano continuum, but there is generally no hope for such a conclusion in Theorem 9, since some components of Z can fail to be locally connected. On the other hand a new technical hurdle (not relevant in [1]) is that Z can have uncountably many components of large diameter, (for example if A is the product of $[0, 1]$ with a Cantor set).

Our proof of Theorem 9 exploits the $CAT(0)$ geometry of PL planar disks as determined by internal paths of minimal Euclidean length. In various settings we wish to select a collection of short disjoint arcs with certain properties. The $CAT(0)$ geometery supplies nontransverse arcs which can be perturbed to be disjoint while satisfying the other desired properties.

2. Definitions and notation

PL denotes **piecewise linear.** An **arc** is a topological space homeomorphic to $[0, 1]$. A **disk** is any space homeomorphic to the closed round unit disk.

If $\alpha \subset R^2$ is an arc then the **length** $l(\alpha)$ is the familiar Euclidean arc length (for example if α is PL then $l(\alpha)$ the sum of the lengths of the finitely many concatenated line segments whose union is α).

Definition 1. *Suppose $A \subset R^2$ is the union of finitely many pairwise disjoint closed PL topological disks. Suppose $B \subset A$ and suppose $B \cap A_i \neq \emptyset$ for each component $A_i \subset A$. Let $N(A, B) = \inf\{\delta > 0|$ for each $x \in \partial A$ there exists $y \in B$ and a PL arc $\gamma \subset A$ connecting x to y such that $l(\gamma) < \delta\}$. Let $M(A, B) = \inf\{\varepsilon > 0|$ for each $x \in A$ there exists $y \in B$ and a PL arc $\gamma \subset A$ connecting x to y such that $l(\gamma) < \delta$.*

The notation **interior** is slightly abused (in the context of arcs and trees) as follows.

If X is a 2 dimensional planar set then $int(X)$ denotes the largest open set $U \subset R^2$ such that $U \subset X$.

If $X \subset R^2$ is a 2 dimensional continuum let $Fr(X) = X\backslash int(X)$ and call $Fr(X)$ the **frontier** of X.

However in the special case $\alpha \subset R^2$ is an arc we let $\partial\alpha$ denotes the endpoints of α and $int(\alpha) = \alpha\backslash\partial\alpha$.

If $E \subset R^2$ is a closed topological disk then ∂E denotes the simple closed curve bounding $int(E)$. Thus $\partial E = \partial(int(E))$.

If $X \subset R^2$ then X is a **tree** if X is connected and simply connected and homeomorphic to the union of finitely many straight Euclidean line segments.

If T is a tree then $int(T) = \{x \in T | T \backslash x \text{ is not connected}\}$ and $\partial T = T \backslash int(T)$.

The tree T is a **triod** if T is homeomorphic to the planar set $([-1,1] \times \{0\}) \cup (\{0\} \times [0,1])$

If X is a tree then x is an **endpoint** of X if $X \backslash \{x\}$ is connected.

If $X \subset R^2$ then X is **cellular** if $R^2 \backslash X$ is connected and simply connected (i.e. X is a nonseparating planar continuum).

The metric space X is a **Peano continuum** if X is compact, connected, and locally path connected.

3. Obtaining nested collections of PL disks S_n

This section clarifies how we approximate a planar compactum X (with connected complement) by nested finite collections of pairwise disjoint PL disks S_n.

Recall definition 1. Informally, using internal path length distance on S_n, M is the Hausdorff distance between S_n and S_{n+1} and N is the Hausdorff distance between $S_{n+1} \cup \partial S_n$ and S_n.

It is immediate $M \geq N$. To see why M and N can be dramatically different, imagine that S_n is a large round disk, and $S_{n+1} \subset S_n$ is a continuum that approximates ∂S_n, but such that S_{n+1} is far from the center of the disk S_n.

Despite the above disparity we have the following Lemma.

Lemma 1. *Suppose $\forall n \geq 1$, $S_n \subset R^2$ is the union of finitely many pairwise disjoint PL closed topological disks such that $S_{n+1} \subset int(S_n)$. Then $\lim_{n\to\infty} M(S_n, S_{n+1}) = 0$ if and only if $N(S_n, S_{n+1}) = 0$*

Proof. Suppose $\lim_{n\to\infty} M(S_n, S_{n+1}) = 0$. Then $\lim_{n\to\infty} N(S_n, S_{n+1}) = 0$ since $\forall n$ we have $N(S_n, S_{n+1}) \leq M(S_n, S_{n+1})$.

Conversely suppose $\lim_{n\to\infty} N(S_n, S_{n+1}) = 0$.

Let $X = \cap_{n=1}^{\infty} S_n$. Note if $x \in S_n$ and if $\varepsilon > 0$ and if $B(x, \varepsilon) \cap (S_{n+1} \cup \partial S_{n+1}) \neq \emptyset$ then there exists a PL path $\gamma \subset S_n$ from x to S_{n+1} whose length is less than $\varepsilon + N(S_n, S_{n+1})$.

To obtain a contradiction suppose $\limsup M(S_n, S_{n+1}) > \varepsilon > 0$. Then, retaining a subsequence and reindexing, there exists $x_n \in S_n$ such that $B(x_n, \varepsilon) \cap (S_{n+1} \cup \partial S_n) = 0$.

Note $B(x_n, \varepsilon) \subset R^2 \backslash X$, since $X \subset S_{n+1}$.

By compactness of S_1, (once again retaining a subsequence and reindexing) we may assume that $x_n \to x$. Note $x \in S_n$ since $\{x_n, x_{n+1}, ..\} \subset S_n$ and S_n is closed. Thus $x \in X$.

On the other hand $|x - x_n| \to 0$. Thus for sufficiently large n, $x \in B(x_n, \varepsilon)$ and thus $x \in R^2 \backslash X$ and we have a contradiction. $\square$

Theorem 1. *Suppose $X \subset R^2$ is compact and $R^2 \backslash X$ is connected. Then there exists a sequence of closed sets $S_n \subset R^2$ such that S_n is the union of finitely many pairwise disjoint closed PL topological disks, such that $S_{n+1} \subset int(S_n)$, such that $X = \cap_{n=1}^{\infty} S_n$ and such that $N(S_n, S_{n+1}) < \frac{1}{10^n}$ and such that*

$$\lim_{n\to\infty} M(S_n, S_{n+1}) = 0.$$

Proof. Let $U = R^2 \backslash X$. Fix $z \in U$. Obtain nested path connected closed sets $A_2 \subset A_3 ...$ such that $U = \cup_{n=2}^{\infty} A_n$ as follows.

Let T_n be a tiling of the plane (such that z is a corner of some tile), by closed squares of sidelength $\frac{1}{2^n}$ parallel to the x or y axis.

Let A_n be a maximal path connected set containing z such that each closed tile of A_n is strictly contained in U. Since U is open and path connected the sets A_n cover U.

Since X is compact obtain $R > 0$ such that $X \subset [-R, R] \times [-R, R]$ and let $S_1 = [-R, R] \times [-R, R]$ and let $A_1 = \emptyset$.

Suppose $n \geq 1$ and S_n has been defined and S_n is a collection of pairwise disjoint topological disks such that $X \subset S_n$ and such that $S_n \cap A_n = \emptyset$.

For each component $P \subset S_n$ and each $x \in P \cap X$ obtain $0 < \delta_x^n < \frac{1}{10^n}$ such that $\overline{B(x, \delta_x^n)} \subset int(P)$ and such that $\overline{B(x, \delta_x^n)} \cap A_{n+1} = \emptyset$. By compactness of $P \cap X$ we obtain a finite subcovering $\{B\{x_i, \delta_{x_i}^n\}\}$. Let $Y_{n+1} = \cup \overline{B(x_i, \delta_{x_i}^n)}$ with components $Q_1, ...Q_m$. By definition for each $y \in Q_i$ there exists $x \in X$ such that the line segment $[y, x] \subset Q_i$ and such that $l([y, x]) < \frac{1}{10^n}$.

Notice Q_i is a locally contractible planar continuum and consequently Q_i has the homotopy type of a disk with finitely many open punctures.

Thicken the outer boundary of Q_i very slightly to obtain pairwise disjoint PL disks $P_1, ..P_m$ such that $Q_i \subset int(P_i)$ and such that $N(P_i, X \cap P_i) < \frac{1}{10^n}$.

Let S_{n+1}denote the union of the PL disks P_i obtained in the fashion just described.

By construction $X \subset S_{n+1} \subset int(S_n)$ and S_{n+1} is the union of finitely many pairwise disjoint P disks such that if P is a component of S_{n+1} then $X \cap P \subset int(P)$. To see that $X = \cap_{n=1}^{\infty} S_n$ it is immediate that $X \subset \cap S_n$ since $X \subset S_n$ for each n. Conversely suppose $y \notin X$. Then there exists $n \geq 2$ such that $y \in A_n$ and in particular $y \notin S_n$.

Since $N(S_n, S_{n+1}) < \frac{1}{10^n}$, it follows that $N(S_n, S_{n+1}) \to 0$ and hence by Lemma 1 $M(S_n, S_{n+1}) \to 0$. □

4. The $CAT(0)$ geometry of a PL planar disk.

If $P \subset R^2$ is a closed PL disk, then internal paths (in P) of minimal Euclidean length determines a metric as follows.

Define $d_P : P \times P \to [0, \infty)$ so that $d(x, x) = 0$ and $\forall M \geq 0$, and $x \neq y$, $d(x, y) \leq M$ iff there exists a PL arc $\alpha \subset P$ such that $\partial \alpha = \{x, y\}$ and $l(\alpha) \leq M$.

Note d_P is a topologically compatible metric (since P is locally path connected and short paths exist locally).

Using polar coordinates, if $R > 0$ and $0 < \psi \leq 2\pi$ let $D(R, \psi) = \{(r, \theta) \in R^2 | r < R \text{ and } \theta < \psi \}$.

(For a careful exposition of the elementary properties of $CAT(0)$ spaces we refer the reader to [2].)

Notice P has a basis of open sets each of which is isometric to some set of the form $D(R, \psi)$, a round Euclidean disk, possibly missing an open sector.

By inspection, pairs of points in $D(R, \psi)$ are connected by a canoniical unique path of minimal length. Moreover the triangles $T \subset D(R, \psi)$ are 'thin', a pair of points in T is at least as close in T as their counterparts in the canonical Euclidean comparison triangle S.

Thus $D(R,\psi)$ is a **CAT(0)** space, and hence (P,d_P) is locally $CAT(0)$. Since P is compact and simply connected, (P,d_p) is $CAT(0)$.

There are 3 important properties of (P,d_P) (which are true in any a $CAT(0)$ space).

1) Given $\{x,y\} \subset P$ there exists a unique arc (or point if $x=y$) of minimal length (a **geodesic**) connecting x and y.

2) The geodesics and their lengths vary continuously with the endpoints.

3) The intersection of two geodesics is connected or empty.

However we will also need the following 4th (special) property of (P,d_P) concerning the geometry of an embedded triod $T \subset P$.

Lemma 2. *Suppose $P \subset R^2$ is a PL disk with $CAT(0)$ metric determined by paths of minimal Euclidean length. Suppose $T \subset P$ is a topological triod such that $\partial T = \{a,b,c\}$ and $x \in T$ is the vertex. Suppose $\alpha_{ab}, \alpha_{bc}, \alpha_{ac}$ denote the arcs in T connecting the respective pairs of endpoints. Then at least one of $\alpha_{ab}, \alpha_{bc}, \alpha_{ac}$ is not a geodesic.*

Proof. The idea is to notice on the small scale near x, T consists of 3 distinct straight segments emanating from x. On the small scale at least one of the 3 complementary sectors must be contained in $int(P)$, and this creates the possibility to shorten the side of T bounding the selected sector.

To obtain a contradiction let $[a,x]\cup[x,b]$, and $[c,x]\cup[x,b]$ and $[c,x]\cup[x,a]$ denote the geodesic sides of T. In particular T is convex in P.

On the other hand we may choose $\varepsilon > 0$ so that if $t < \varepsilon$ we have 3 distinct Euclidean line segments emanating from $x \in P : [x,ta] \subset [x,a]$, $[x,tb] \subset [x,b]$, and $[x,tc] \subset [x,c]$.

Since $P \subset R^2$ is a PL disk, there exists $\delta < \varepsilon$, such that $\overline{B(x,\delta)} \cap P$ is isometric to a round disk with an open (possibly empty) sector missing (and we allow that a sector can have angle $> 180^0$). Note $\overline{B(x,\delta)} \cap P$ is convex in P.

However, by inspection $T \cap \overline{B(x,\delta)}$ is not convex in $\overline{B(x,\delta)} \cap P$ contradicting our assumption that T is convex in P. $\square$

5. Perturbing a PL disk

Given a PL disk $Q \subset R^2$, with finitely many marked points $Y \subset \partial Q$ we wish to show the existence of arbitrarily small perturbations P of Q, so that the respective geometries of P and Q are very close to each other, and so that $P \subset Q$ and $Q \cap \partial P = Y$.

This is obvious and the idea is simply to push the components of $\partial Q \backslash Y$ inward by a very tiny amount.

Lemma 3. *Suppose $\alpha \subset R^2$ is a PL arc and $\delta > 0$. There exist PL arcs $\beta \subset R^2$ and $\gamma \subset R^2$ such that $int(\beta) \cap int(\gamma) = \emptyset$ and α is a spanning arc of the PL disk $D(\beta,\gamma)$ and there exists a homeomorphism $h : D(\beta,\gamma) \to D(\beta,\alpha)$ such that $\left|d_{D(\beta,\alpha)}(h(x),h(y)) - d_{D(\beta,\gamma)}(x,y)\right| < \delta$.*

Proof. Let $v_0, ..., v_n$ denote consecutive vertices of α. Notice if γ is sufficiently small there exist points $w_1, ...w_{n-1}$ such that $w_i \notin \alpha$ and $|w_i - v_i| < \gamma$ and such that all of the following hold:

1) The PL path $\beta = [w_0,w_1]\cup[w_1,w_2]...\cup[w_{n-1},w_n]$ is an arc such that $int(\alpha)\cap int(\beta) = \emptyset$.

2) If T_i denotes the convex hull of $\{v_{i-1}, w_{i-1}, v_i, w_i\}$, then T_i is convex in R^2.

3) The closed disk $D(\beta,\alpha)$ bounded by α and β can be canonically fibred by line segments as follows:

If $L_i : [v_i, v_{i+1}] \to [w_i, w_{i+1}]$ is the order preserving linear homeomorphism then $[v_{i,t}, L_i(v_{i,t})] \subset \dot{D(\beta,\alpha)}$

and if $j \neq i$ or if $s \neq t$ then $[v_{i,t}, L_i(v_{i,t})] \cap [v_{j,s}, L_i(v_{j,s})] = \emptyset$.

4) If $i \in \{2, 3, ...n-1\}$ then $D(\beta,\alpha)\backslash T_i$ is disconnected.

Given α and β, we can perform a similar construction on the other side of α to obtain an arc γ with the same properties (with respect to α) as β such that $int(\gamma) \cap int(\beta) = \emptyset$.

Let $S_1 < .. < S_n$ denote the convex cells of $D(\alpha,\gamma)$ and let $u_0, ..u_n$ denote the vertices of γ.

Now from the above construction we have canonical PL homeomorphisms $f : \alpha \to \beta$ and $g : \alpha \to \gamma$.

Hence we obtain a canonical homeomorphism $h : D(\beta,\gamma) \to D(\alpha,\gamma)$ as follows. For each $x \in \alpha$, let h map the PL arc $[f(x), x] \cup [x, g(x)]$ 'linearly' onto the segment $[x, g(x)]$ so that $h^{-1}_{[x,g(x)]}$ has constant speed.

Q Since T_i is convex, if the line segment $\gamma_i \subset T_i$ connects $[w_{i-1}, v_{i-1}]$ to $[v_i, w_i,]$ then $l([w_{i-1}, v_{i-1}]) \leq l(\gamma_i) \leq l([v_i, w_i,])$ or $l([w_{i-1}, v_{i-1}]) \geq l(\gamma_i) \geq l([v_i, w_i,])$.

If γ is small then $||w_{i-1} - w_i| - |v_{i-1} - v_i|| < \frac{\delta}{2n}$ and $||u_{i-1} - u_i| - |v_{i-1} - v_i|| < \frac{\delta}{2n}$

By condition 4, a given geodesic $\lambda \subset \dot{D(\beta,\gamma)}$ is the union of line segments from consecutive cells $T_i \cup S_i, ..., T_{i+k} \cup S_{i+k}$.

Consequently $l(h(\gamma)) < l(\lambda) + n(\frac{\delta}{2n} + \frac{\delta}{2n}) = l(\lambda) + \delta$.

In similar fashion a given geodesic $\lambda \subset \dot{D(\alpha,\gamma)}$ is the union of line segments from consecutive cells $S_i, S_{i+1}, ...S_{i+k}$. Hence $l(h^{-1}(\lambda) < l(\lambda) + \delta$. □

Lemma 4. *Suppose $P \subset R^2$ is a PL disk and $Y = \{y_1.., y_n\} \subset \partial P$ and $\varepsilon > 0$. There exists a PL disk $Q \subset R^2$ such that $P \subset Q$ and $\partial P \cap \partial Q = Y$ and there exists a homeomorphism $h : Q \to P$ such that $|d_Q(x,y) - d_P(h(x), h(y))| < \varepsilon$ and such that $h_Y = id_Y$.*

Proof. Let $\alpha_1, ...\alpha_n$ denote the arcs between consecutive points of Y so that $\cup\alpha_i = \partial P$. Let A denote the set of vertices of ∂P and let $M = |A \cup Y|$.

Apply the previous Lemma with $\delta = \frac{\varepsilon}{M}$ to each α_i so that $int(D(\beta_i, \gamma_i)) \cap int(D(\beta_j, \gamma_j)) = \emptyset$ if $i \neq j$.

Let $Q = \cup D(\beta_i, \alpha_i) \cup P$ and let $h_i : D(\beta_i, \gamma_i) \to D(\alpha_i, \gamma_i)$ be the canonical homeomorphism (defined in Lemma 3) and let E denote the PL disk bounded by $\cup\gamma_i$.

Notice we have a total of $2M$ sets of the form T^i_j or S^i_j and each set T^i_j and each set S^i_j is a convex Euclidean quadrilateral.

In particular if $\lambda \subset Q$ is a geodesic then $\gamma \cap T^i_j$ is a (connected) geodesic and $\gamma \cap S^i_j$ is a (connected) geodesic and thus $l(h(\gamma \cap T^i_j)) < l(\gamma \cap T^i_j) + \frac{\varepsilon}{2M}$ and $l(h(\gamma \cap S^i_j)) < l(\gamma \cap S^i_j) + \frac{\varepsilon}{2M}$

In similar fashion if $\lambda \subset P$ is a geodesic then $\lambda \cap S^i_j$ is a (connected) geodesic and $l(h^{-1}(\gamma \cap S^i_j)) < l(\gamma \cap S^i_j) + \frac{\varepsilon}{2M}$.

Thus if $\lambda \subset Q$ is a geodesic we let $\lambda = \psi_1 \cup \psi_2....$ such that $\psi_1, \psi_2, ...$ are consecutive distinct components of either $\lambda \cap E$ or $\lambda \cap (\cup_i D(\beta_i, \gamma_i))$.

Note if ψ_j is a component of $\lambda \cap E$ then $h(\psi_j) = \psi_j$ and hence $l(\psi_j) = l(h(\psi_j))$

If ψ is a component of $\lambda \cap D(\beta_i, \gamma_i)$ then $l(h(\psi_j)) < l(\psi_j) + \frac{\varepsilon}{M}$ and thus $l(h(\lambda)) = \Sigma l(h(\psi_i)) + \varepsilon$

In similar fashion if $\lambda \subset P$ is a geodesic we let $\lambda = \psi_1 \cup \psi_2 ....$ such that $\psi_1, \psi_2, ...$ are consecutive distinct components of either $\lambda \cap E$ or $\lambda \cap (\cup_i D(\alpha_i, \gamma_i))$.

Note if ψ_j is a component of $\lambda \cap E$ then $h^{-1}(\psi_j) = \psi_j$ and hence $l(\psi_j) = l(h^{-1}(\psi_j))$

If ψ_j is a component of $\lambda \cap D(\alpha_i, \gamma_i)$ then $l(h^{-1}(\psi_j)) < l(\psi_j) + \frac{\varepsilon}{M}$ and thus $l(h^{-1}(\lambda)) = \Sigma l(h(\psi_i)) + \varepsilon$. □

Theorem 2. *Suppose $Q \subset R^2$ is a PL disk and $Y \subset \partial Q$ is finite. Suppose $E \subset int(Q) \cup Y$ and assume E is a PL compactum with finitely many components, and assume $\varepsilon > 0$. Then there exists a PL disk $P \subset Q$ such that $P \cap \partial Q = Y$ and there exists a homeomorphism $h : Q \to P$ such that if d_P and d_Q denote the respective $CAT(0)$ metrics on P and Q then $h_E = id_E$, and if $\{x, y\} \subset Q$ then $|d_P(h(x), h(y)) - d_Q(x, y)| < \varepsilon$.*

Proof. Starting with Q, push the components of $\partial Q \backslash Y$ inward by a sufficiently tiny amount, obtaining a PL disk $P \subset Q$ so that so that, recalling Lemma 4, the homeomorphism $h : Q \to P$ fixes $E \cup Y$ pointwise. □

6. Connecting ∂S_n to S_n with very short arcs

Recall we have $S_{n+1} \subset S_n$ and S_i is a collection of pairwise disjoint PL disks in R^2 such that $S_{n+1} \subset int(S_n)$ and each component of S_n contains at least one component of S_{n+1}.

Given finitely many marked points $Y_n \subset \partial S_n$, we wish naively to connect Y_n to ∂S_{n+1} with pairwise disjoint arcs in S_n of minimal length. The simplest strategy almost works. If we select the shortest possible arcs while ignoring the constraint of disjointedness, the $CAT(0)$ geometry guarantees the selected arcs don't cross transversely. Consequently we can perturb (arbitrarily) the selected arcs to be disjoint, (and in our particular application all will have length less than $\frac{1}{10^n}$).

Lemma 5. *Suppose $\alpha_1, ..\alpha_n$ are PL arcs in the plane with common endpoint z such that $\alpha_i \cap \alpha_j$ is connected. Then $\cup_{i=1}^n \alpha_i$ is a tree.*

Proof. Note α_1 is simply connected. By induction suppose $\cup_{i=1}^{k-1} \alpha_i$ is simply connected. Let $\partial \alpha_k = \{x, z\}$. If $x \in \cup_{i=1}^{k-1} \alpha_i$ then let $x \in \alpha_i$ for $i \leq k-1$. Thus $\alpha_k \subset \alpha_i$ since, by the hypothesis of the Lemma, $\alpha_k \cap \alpha_i$ is connected. Hence $\cup_{i=1}^{k-1} \alpha_i = \cup_{i=1}^{k} \alpha_i$ and by the induction hypothesis both spaces are simply connected.

If $x \notin \cup_{i=1}^{k-1} \alpha_i$ let J be the component of $(\cup_{i=1}^{k-1} \alpha_i) \backslash \alpha_k$ such that $x \in J$. Let $y = \overline{J} \backslash J$. Let $y \in \alpha_i$ for $i < k$. Then $(\alpha_k \backslash J) \subset \alpha_i$ since $\{y, z\} \subset \alpha_k \cap \alpha_i$.

To obtain a contradiction suppose $\cup_{i=1}^{k} \alpha_i$ is not simply connected. Let $S \subset \cup_{i=1}^{k} \alpha_i$ be a simple closed curve. By the induction hypothesis $\cup_{i=1}^{k-1} \alpha_i$ is simply connected and hence $\alpha_k \cap S \neq \emptyset$ and hence $S \cap J \neq \emptyset$. Now we have a contradiction since J is an initial segment of α_k and $J \cap (\cup_{i=1}^{k-1} \alpha_i) = \emptyset$. □

Theorem 3. *Suppose each of the sets $S_n \subset R^2$ and $S_{n+1} \subset R^2$ is a collection finitely many pairwise disjoint closed PL disks. Suppose $S_{n+1} \subset int(S_n)$ and $N(S_n, S_{n+1}) < \delta_n$. Suppose $Y_n = \{y_1, , ., y_m\} \subset \partial S_n$. Then there exists a collection of pairwise disjoint closed arcs $\gamma_1, ..\gamma_m$ such that $int(\gamma_i) \subset int(S_n) \backslash S_{n+1}$ and γ_i connects y_i to S_{n+1} and $l(\gamma_i) < \delta_n$.*

Proof. The strategy is to first recursively select arcs of minimal length $\alpha_1,..\alpha_m$ in S_n such that α_i connects y_i to S_{n+1}. Unfortunately it can happen that $\alpha_i \cap \alpha_j \neq \emptyset$. However, by virtue of our construction, if $\alpha_i \cap \alpha_j \neq \emptyset$ then $\alpha_i \cap \alpha_j$ is a final segment of each of α_i and α_j. Consequently we can perturb the arcs $\{\alpha_i\}$ to very short pairwise disjoint arcs $\{\gamma_i\}$.

Recall each component of S_n admits a canonical $CAT(0)$ metric determined by minimal Euclidean path length between points.

By compactness of S_n, there exists an arc $\alpha_1 \subset S_n$ of minimal length connecting y_1 to ∂S_{n+1}. Let $\partial(\alpha_1) = \{y_1, z_1\}$. We proceed recursively as follows.

Suppose the geodesic arcs $\alpha_1, , \alpha_{i-1} \subset S_n$ have been chosen so that if $k \leq i-1$ then α_k is a path of minimal length connecting y_k to ∂S_{n+1} at z_k and suppose if $k < j \leq i-1$ and if $\alpha_k \cap \alpha_j \neq \emptyset$ then $z_k = z_j$.

Let α_i^* be a minimal arc connecting y_i to ∂S_{n+1} in S_n. If $\alpha_i^* \cap \alpha_j = \emptyset$ for all $j < i$ then let $\alpha_i = \alpha_i^*$ and notice the induction hypothesis holds for $\{\alpha_1, .., \alpha_i\}$.

If there exists $j < i$ such that $\alpha_i^* \cap \alpha_j \neq \emptyset$ let $\alpha_i^* \cap \alpha_j = [x, y]$ with $y_j \leq x \leq y \leq z_j$. Notice $l([x, z_i]) = l([x, z_j])$ since otherwise one of α_j or α_i could be strictly shortened contradicting minimality. Define $\alpha_i = [y_i, x] \cup [x, z_j]$. Note $l(\alpha_i^*) = l(\alpha_i)$ and hence α_i has minimal length. If $\alpha_i \cap \alpha_k \neq \emptyset$ then, by definition of α_i and the induction hypothesis we have $z_i = z_j = z_k$.

Finally, note since $N(S_n, S_{n+1}) < \delta_n$, that $l(\alpha_i) < \delta_n$ for all i.

To perturb the arcs $\{\alpha_i\}$, first notice the collection $\{\alpha_i\}$ is naturally partitioned via the equivalence relation $\alpha_i\tilde{}\alpha_j$ if and only if $z_i = z_k$.

Thus for each equivalence class $[z_i]$ consider the arcs $\beta_1^i, ..., \beta_k^i \subset \{\alpha_1, , , \alpha_m\}$ such that β_j^i connects y_j^i to z_i. Note if $z_k \neq z_i$ then $(\cup_j \beta_j^i) \cap (\cup_j \beta_j^k) = \emptyset$ since if $z_i \neq z_k$ then $\alpha_i \cap \alpha_k = \emptyset$.

Fixing i, note $\{y_j^i\}$ inherits a canonical circular order from the simple closed curve component of ∂S_n. Now we will select a 'starting point' $y \in \{y_j^i\}$ as follows.

Let D_i denote the component of S_{n+1} such that $z_i \in D_i$. Notice $z_i \in \partial D_i$ since β_j^i has minimal length.

Let $T^i = \cup \beta_j^i$. By Lemma 5 T^i is a finite PL tree, and, by hypothesis, if $k \neq i$ then $T^i \cap T^k = \emptyset$.

Fix an arbitrary point $w \in \partial D_i \backslash z_i$. Start at w and travel clockwise along ∂D_i and stop at z_i. Then travel monotonically along T^i, turning 'left' whenever possible and stop at $y \in Y_n$, and we have canonically obtained a 'starting point' from our circularly ordered set $\{y_j^i\}$.

Now, keeping i fixed and permuting j, reindex $\{\beta_j^i\}$such that $y = y_1^i < ... < y_m^i$, ordered in clockwise fashion on ∂S_n.

Theorem 2 ensures we can obtain arbitrarily small perturbations of $\{\beta_j^i\}$ to obtained the desired arcs $\{\gamma_j^i\}$. □

7. Connecting cellular sets in PL disks with short arcs

Recall at the nth stage of our construction we have $S_{n+1} \subset S_n$ and S_i is a collection of pairwise disjoint PL disks such that $S_{n+1} \subset int(S_n)$ and each component of S_n contains at least one component of S_{n+1}. At this stage we also have finitely many arcs pairwise disjoint PL arcs $\gamma_1, \gamma_2, ...$ connecting ∂S_n to S_{n+1} such that $l(\gamma_i) < \frac{1}{10^n}$ and such that $int(\gamma_i) \subset S_n \backslash S_{n+1}$.

Note each component $D_j \subset S_{n+1} \cup \gamma_1 \cup \gamma_2 ...$ is a PL cellular set.

Our naive hope is to attach closed disjoint arcs $\alpha_1, \alpha_2, ,$,of minimal length to $\cup D_j$ in order create one cellular continuum in each component of S_n, and we also hope that $\alpha_i \cap \gamma_j = \emptyset$.

The simplest strategy almost works. If we begin connecting together the sets $D_1, D_2, ..$ with minimal length arcs $\alpha_1, \alpha_2, ..$, without regard to whether the newly selected arcs $\{\alpha_n\}$ are disjoint, ultimately the $CAT(0)$ structure on the components of S_n ensures our newly selected arcs do not cross transversely.

However there are two technical problems with the output arcs $\alpha_1, \alpha_2, ..$

As mentioned, the first problem is the arcs $\{\alpha_n\}$ are typically not disjoint, however this can be fixed by arbitrarily small perturbations, since the arcs $\{\alpha_n\}$ do not cross each other transversely.

The second problem is that $\alpha_i \cap \gamma_j \neq \emptyset$ can happen, yet we were hoping that $\alpha_i \cap \gamma_j = \emptyset$ for all i and j. This problem can be fixed by a perturbation on the order of $\frac{1}{10^n}$, since $l(\gamma_i) < \frac{1}{10^n}$, and the idea is to 'slide' the arcs α_i off of γ_j (in a tiny neighborhood of γ_j) so that $\alpha_i \subset S_n$ connects distinct PL disks from S_{n+1}.

By construction the originally selected arcs $\alpha_1, \alpha_2, ...$are 'short' and have length at most $2M(S_n, S_{n+1})$ (double the Hausdorff (using path length) distance between S_n, S_{n+1}).

After two or three perturbations of the arcs $\{\alpha_n\}$ we have arcs $\{\alpha_n^{***}\}$ such that $l(\alpha_n^{***}) < 2M(S_n, S_{n+1}) + 2N(S_n, S_{n+1})$ with all the desired properties, namely α_n^{***} connects distinct components of S_{n+1} and $int(\alpha_n^{***}) \subset int(S_n)$, and $\alpha_i^{***} \cap \gamma_j = \emptyset$ for all i and j.

Combining all the constructions and perturbations in this section 7 (and its subsections), we obtain the following theorem, ultimately critical to the recursive process by which we will attach a null sequence of arcs to an arbitrary planar compactum.

Theorem 4. *Suppose $P \subset R^2$ is a closed PL disk and $E_1, E_2, .., E_n$ are pairwise disjoint closed PL disks such that $\cup E_i \subset int(P)$. Suppose $\{y_1, ..., y_m\} \subset P$ and suppose $\gamma_1, \gamma_2, .., \gamma_m$ is a collection of pairwise disjoint closed PL arcs such that γ_i connects y_i to $\partial(\cup E_i)$ and such that $l(\gamma_i) < N(P, \cup E_i)$ and such that $int(\gamma_i) \subset P \backslash (\cup E_i)$. Then there exist finitely many pairwise disjoint PL closed arcs $\alpha_1^{***}, \alpha_2^{***}, ... \subset P$, such that $l(\alpha_i^{***}) < 2(M(P, \cup E_i) + N(P, \cup E_i))$ and such that α_i^{***} connects distinct components of $\cup E_i$, such that $int(\alpha_i^{***}) \subset P \backslash ((\cup E_i) \cup (\cup \gamma_j))$, and such that $\{\cup \alpha_i^{***}\} \cup \{\cup \gamma_j\} \cup \{\cup E_k\}$ is cellular.*

7.1. **An arc selection algorithm.** The **input** for the algorithm is the data $P, D_1, ..D_n$ satisfying the following two conditions:

1) $P \subset R^2$ is a closed PL disk.

2) $\{D_i\}$ is a collection of pairwise disjoint cellular sets such that $\cup_{i=1}^n D_i \subset P$.

Consider the topologically compatible $CAT(0)$ metric $d : P \times P \to [0, \infty)$ satisfying $\forall M \geq 0$, $d(x, y) \leq M$ iff there exists a PL path in P connecting x to y of Euclidean pathlength M or less.

Starting at $k = 1$ select arcs $\alpha_k \subset P$ recursively as follows.

Let $F_k = D_1 \cup D_2 .. \cup D_n \cup \alpha_1 ... \alpha_{k-1}$.

If F_k is connected terminate the algorithm.

If F_k is not connected, let C_k denote the set of all paths β in P such that β connects distinct components of F_k and such that the endpoints of β belong to distinct components of $\cup_{i=1}^n D_i$.

Let P_k denote the set of all $g \in C_k$ such that g has minimal length. Note $P_k \neq \emptyset$ since F_k and $\cup_{j=1}^{n} D_j$ are compact.

If possible, select $\alpha_k \in P_k$ such that for all $i < k$, $\alpha_i \cap \alpha_k$ does not disconnect α_k. Otherwise terminate the algorithm.

The algorithm eventually terminates since F_0 has finitely many components and F_{k-1} has strictly fewer components than F_k.

Lemma 6. *Suppose $\alpha_1, ..\alpha_k$ have been selected by the previous algorithm. Suppose $M(P, \cup D_i) < \varepsilon$. Then for each i, $l(\alpha_i) < 2\varepsilon$ and $int(\alpha_i) \subset P \backslash (\cup_{k=1}^{n} D_k)$. If $i < k$ then $l(\alpha_i) \leq l(\alpha_k)$.*

Proof. Suppose A and B is any separation of $\cup_{j=1}^{n} D_j$. Since A and B are compact, there exists a path of minimal length g connecting A to B and such a path must be a geodesic arc (since all nongeodesics in P can be strictly shortened while keeping the endpoints fixed), and since all nontrivial geodesics in a $CAT(0)$ space are topological arcs.

Let x be the midpoint of g and let $g = [a, x] \cup [x, b]$ with $l([a, x]) = l([x, b])$ and $a \in A$ and $b \in B$.

Let $[x, z]$ be a geodesic connecting x to $A \cup B$ such that $l([x, z]) < \varepsilon$. Since $z \in A \cup B$ wolog we may assume $z \in A$. To obtain a contradiction assume $l([z, x]) < l([a, x])$. Then the path $[z, x] \cup [x, b]$ connects A to B and $l([z, x] \cup [x, b]) < l(g)$ contradicting the fact that g has minimal length among all such paths. Thus $l([z, x]) = l([a, x])$ and hence $l(g) < 2\varepsilon$.

By definition α_i connects distinct components E and G of F_i. Let $A = E \cap (\cup_{k=1}^{n} D_k)$ and let $B = (F_i \backslash E) \cap (\cup_{k=1}^{n} D_k)$. It follows that α_i is an arc of minimal length connecting A and B. Thus $l(\alpha_i) < 2\varepsilon$.

Let $\beta_i : [0, 1] \to \alpha_i$ be a homeomorphism such that $\beta_i(0) \in A$.

To prove $int(\alpha_i) \subset P \backslash (\cup D_i)$, let t be maximal such that $\beta_i(t) \in A$. It follows that there exists $\delta > 0$ such that if $s \in (t, t + \delta)$ then $\beta_i(s) \in \alpha_i \cap (P \backslash (A \cup B))$.

Let J the component of $\alpha_i \cap (P \backslash (A \cup B))$ such that $\beta_i(t, t + \delta)) \subset J$. It follows that the other endpoint of J is in B.

Then $J = int(\alpha_i)$, (since otherwise $l(J) < l(\alpha_i)$ contradicting the fact that $l(\alpha_i)$ is minimal among arcs in P connecting A and B.

Notice if $i < k$ then $C_k \subset C_i$ and hence $l(\alpha_i) \leq l(\alpha_k)$. □

7.1.1. *If $k \leq n-1$ then α_k exists.* It is not obvious when our algorithm terminates. In principle two selected arcs α_i and α_j could cross transversely and thus terminate the algorithm prematurely.

However, the Theorem in this section shows the aforementioned disaster does not occur.

In the proof we break the hypothetical data into cases and we argue the least obvious case first, and we exploit symmetry of the data to cut down the number of cases to a manageable size.

Lemma 2 is of particular importance in the least obvious cases.

Theorem 5. *For each $k < n$ there exists $\alpha_k \in P_k$ such that for all $i < k$ $\alpha_i \cap \alpha_k$ does not disconnect α_k.*

Proof. If $n = 1$ the theorem is vacuously true. Suppose $n \geq 2$. Notice α_1 exists.

To obtain a contradiction assume the Theorem at hand is false. Choose $k < n-1$ minimal so that the theorem false.

Thus for all $\beta \in P_k$ there exists $j < k$ such that $\alpha_j \cap \beta$ disconnects β.

Choose i maximal so that there exists $\beta \in P_k$ such that $\beta \cap \alpha_j$ does not disconnect β for all $j < i$.

Now define $\alpha_k \in P_k$ such that $\alpha_k \cap \alpha_j$ does not disconnect α_k for all $j < i$ and such that $\alpha_k \cap \alpha_i$ disconnects α_k.

Note $\alpha_i \cap \alpha_k \neq \emptyset$ (since otherwise α_i and α_k are disjoint and in particular $\alpha_i \cap \alpha_k$ would fail to disconnect α_k.)

Since $i \neq k$, $\alpha_i \neq \alpha_k$ (since α_k connects distinct components of F_k, and the continuum α_i is contained in some component of F_k). Hence $\partial\alpha_i \neq \partial\alpha_k$ by uniqueness of geodesics with common endpoints.

Thus $3 \leq |\partial\alpha_i \cup \partial\alpha_k| \leq 4$. We show $|\partial\alpha_i \cup \partial\alpha_k| = 4$ as follows.

(If $3 = |\partial\alpha_i \cup \partial\alpha_k|$ let $\{b\} = \partial\alpha_i \cap \partial\alpha_k$. Thus $b \in \alpha_i \cap \alpha_k$. Moreover $\alpha_i \cap \alpha_k$ is a geodesic since each of α_i and α_k is a geodesic. In particular $\alpha_i \cap \alpha_k$ is connected. Thus, since $b \in \alpha_i \cap \alpha_k$, $\alpha_i \cap \alpha_k$ is an initial segment of α_k, contradicting our assumption that $\alpha_k \cap \alpha_i$ disconnects α_k.)

Since $|\partial\alpha_i \cup \partial\alpha_k| = 4$, $(\alpha_i \cap \alpha_k) \subset int(\alpha_k)$ and $(\alpha_i \cap \alpha_k) \subset int(\alpha_i)$.

Let $[z, y] = int(\alpha_k) \cap int(\alpha_i)$ and let $\alpha_i = [a, z] \cup [z, y] \cup [y, b]$ and let $\alpha_k = [c, z] \cup [z, y] \cup [y, d]$.

Let $x \in [z, y]$.Thus we have five distinct points $\{a, b, c, d, x\}$. Moreover we see that each of $\{a, x, b, c\}$ is $\{a, x, b, d\}$ noncolinear as follows.

(By symmetry it suffices to see that $c \cup \alpha_i$ is not colinear. Let α be a geodesic containing α_i. Recall $c \notin [a, b]$ and thus if $c \in \alpha_i$ then wolog $c < a < x$ on α. However $a \notin [c, x]$ and we have a contradiction.).

We begin with the hardest cases.

A: The cases $l[a, x] = l[b, x] = l[c, x]$ or $l[a, x] = l[b, x] = l[d, x]$

By symmetry it suffices to treat the case $l[a, x] = l[b, x] = l[c, x]$.

Let G and H denote the components of F_i such that $a \in G$ and $b \in H$.

Case A1. If $c \notin G \cup H$ then $\{[a, x] \cup [x, c], [c, x] \cup [x, b]\} \subset C_i$ and each have length equal to that of α_i.

By definition $[a, x] \cup [x, b]$ is a geodesic. By Lemma 2 , at least one of $[a, x] \cup [x, c]$ of $[c, x] \cup [x, b]$ is not a geodesic and can be strictly shortened while keeping the endpoints fixed, contradicting our choice of α_i.

Case A2. Suppose $c \in G$. Then $c \notin H$. Let $\hat{\alpha_k} = [a, x] \cup [x, d]$. Note $\hat{\alpha_k} \in C_k$ and $l(\hat{\alpha_k}) = l(\alpha_k)$. If $\hat{\alpha_k}$ is not a geodesic then it can be strictly shortened while keeping the endpoints fixed, contradicting our choice of α_k. Thus we may assume that $\hat{\alpha_k}$ is also a geodesic.

Suppose $j < i$.

Case A2a. Suppose $x \notin \alpha_j \cap \hat{\alpha_k}$.

Then $(\hat{\alpha_k} \cap \alpha_j) \subset [a, x]$ or $(\hat{\alpha_k} \cap \alpha_j) \subset [x, d]$.

If $(\hat{\alpha_k} \cap \alpha_j) \subset [a, x]$ then $\hat{\alpha_k} \cap \alpha_j = \alpha_i \cap \alpha_j$ which is an initial segment of $\alpha_i = [a, x] \cup [x, b]$ by induction hypothesis. Hence $a \in (\hat{\alpha_k} \cap \alpha_j)$ and in particular $(\hat{\alpha_k} \cap \alpha_j)$ does not separate $\hat{\alpha_k}$.

If $(\hat{\alpha_k} \cap \alpha_j) \subset [d, x]$ then $\hat{\alpha_k} \cap \alpha_j = \alpha_k \cap \alpha_j$ which is an initial segment of $\alpha_k = [c, x] \cup [x, d]$ by the induction hypothesis. Hence $d \in (\hat{\alpha_k} \cap \alpha_j)$ and in particular $(\hat{\alpha_k} \cap \alpha_j)$ does not separate $\hat{\alpha_k}$.

Case A2b. Suppose $x \in \alpha_j \cap \hat{\alpha_k}$. Note $x \in \alpha_i \cap \alpha_k \cap \alpha_j$.

Since $int(\alpha_j) \cap int(\alpha_i) \neq \emptyset$, by induction hypothesis $\alpha_j \cap \alpha_i$ is an initial segment of α_i.

If $a \in \alpha_j \cap \alpha_i$ then it follows that it follows that $\hat{\alpha_k} \cap \alpha_j$ is an initial segment of $\hat{\alpha_k}$ and hence $\hat{\alpha_k} \cap \alpha_j$ does not disconnect $\hat{\alpha_k}$.

Thus we may assume henceforth that $b \in \alpha_j \cap \alpha_i$. It follows that $[b, x] \subset (\alpha_j \cap \alpha_i)$ since $\alpha_j \cap \alpha_i$ is a geodesic containing $\{x, b\}$

Since $int(\alpha_j) \cap int(\alpha_k) \neq \emptyset$, by induction hypothesis $\alpha_j \cap \alpha_k$ is an initial segment of α_k.

If $c \in \alpha_j \cap \alpha_k$ then it follows that $\hat{\alpha_k} \cap \alpha_j$ is an initial segment of $\hat{\alpha_k}$ and hence $\hat{\alpha_k} \cap \alpha_j$ does not disconnect $\hat{\alpha_k}$.

Thus we may assume henceforth that $d \in \alpha_j \cap \alpha_k$. It follows that $[d, x] \subset (\alpha_j \cap \alpha_k)$ since $\alpha_j \cap \alpha_k$ is a geodesic containing $\{x, d\}$

By Lemma 2, since $[a, x] \cup [x, d]$ and $[a, x] \cup [x, b]$ are geodesics, the path $[b, x] \cup [x, d]$ is not a geodesic.

On the other hand the nongeodesic $[b, x] \cup [x, d]$ is a subarc of the geodesic α_j and we have a contradiction.

Conclusion of case A2. We have shown that $\hat{\alpha_k} \in P_k$ and for all $j \leq k-1$ we have that $\hat{\alpha_k} \cap \alpha_j$ does not disconnect $\hat{\alpha_k}$. This contradicts our original assumptions on k and i.

Case A3. Suppose $c \notin G$ and $c \in H$. Then repeat case 2 while exchanging the roles of a and b.

For the remaining cases, by symmetry, we lose no generality in assuming that $l([a, x]) \leq l([b, x])$.

B: The cases $l[c, x] < l[a, x]$ or $l[d, x] < l[a, x]$.

By symmetry we may assume $l[c, x] < l[a, x]$.

Note $l[c, x] < l([b, x])$. Recall $\alpha_i = [a, x] \cup [x, b]$.

Let G and H be distinct components of F_i such that $a \in G$ and $b \in H$. Since $G \cap H = \emptyset$, $c \notin G \cap H$.

If $c \notin G$ then $[a, x] \cup [x, c] \in C_i$ and $l([a, x] \cup [x, c]) < l(\alpha_i)$ contradicting our choice of α_i.

If $c \in G$ then $c \notin H$. Note $[c, x] \cup [x, b] \in C_i$ and $l([c, x] \cup [x, b]) < l(\alpha_i)$ and again we have a contradiction.

C: The case $l[c, x] > l[a, x]$ and $l[d, x] > l[a, x]$

Recall $\alpha_k = [c, x] \cup [x, d]$.

Let G and H be distinct components of F_k such that $c \in G$ and $d \in H$. Since $G \cap H = \emptyset$, $a \notin G \cap H$.

If $a \notin G$ then $[a, x] \cup [x, c] \in C_k$ and $l([a, x] \cup [x, c]) < l(\alpha_k)$ contradicting our choice of α_k.

If $a \in G$ then $a \notin H$. Note $[a, x] \cup [x, d] \in C_k$ and $l([a, x] \cup [x, d]) < l(\alpha_k)$ and again we have a contradiction.

D: The cases $(l[a, x] = l[c, x] \leq l[x, d])$ or $([a, x] = l[d, x] \leq l[x, c])$.

By symmetry we assume $l[a, x] = l[c, x] \leq l[x, d]$. If $l[a, x] = l[b, x]$ then we have treated this case already.

Thus we may assume $l[a, x] < l[b, x]$.

It follows that $l[c, x] < l[d, x]$ since otherwise $l[b, x] = l[a, x]$ (since $l(\alpha_i) \leq l(\alpha_k)$).

Thus we may assume $l[c, x] < l([d, x])$. Let G be the component of F_i such that $a \in G$.

Suppose $c \notin G$. Then $[a,x] \cup [x,c] \in C_i$ and $l([a,x] \cup [x,c]) < l(\alpha_i)$ contradicting our choice of α_i.

Suppose $c \in G$. Then $[a,x] \cup [x,d] \in C_k$ and $l([a,x] \cup [x,d]) < l(\alpha_k)$ contradicting our choice of α_k. □

Remark 1. *By construction F_k has $n-k+1$ components, and since α_{n-1} exists $F_n = D_1 \cup .. \cup D_n \cup \alpha_1 .. \cup \alpha_{n-1}$ is connected.*

7.2. **Perturbing the arcs $\{\alpha_i\}$.** Appealing to section 7.1, starting with a closed topological PL disk $P \subset R^2$ and pairwise disjoint closed PL cellular sets $D_1, ..D_n$ (such that $D_n \subset P$) we have obtained a sequence of closed arcs $\alpha_1, , ...\alpha_{n-1}$ such that $\partial\alpha_n \subset \cup(\partial D_i)$ and $int(\alpha_i) \subset P \backslash (\cup D_i)$ and such that $D_1 \cup .. \cup D_n \cup \alpha_1 .. \cup \alpha_{n-1}$ is cellular (since if A and B are disjoint cellular planar continua and $\beta \subset R^2$ is a closed arc such that $\partial\alpha \subset A \cup B$ and $int(\alpha) \subset R^2(A \cup B)$ and if $A \cup \alpha \cup \beta$ is connected then $A \cup \alpha \cup B$ is cellular). It is also the case that $\alpha_i \cap \alpha_j$ is a closed initial segment of α_i for all i and j.

7.2.1. *Perturbing the interiors of $\{\alpha_i\}$.* Our first task is to perturb the interiors of the open arcs $int(\alpha_i)$ to be disjoint, and we need the following Lemma.

Lemma 7. *For each i there exists $z_i \in int(\alpha_i)$ such that $z_i \notin \alpha_j$ for all $j \neq i$.*

Proof. To obtain a contradiction suppose the Lemma is false. For some endpoint $a \in \alpha_i$, there exist $j \neq i$ such that $\alpha_i \cap \alpha_j \neq \emptyset$ and such that $a \in \alpha_i \cap \alpha_j$. Choose j such that $\alpha_i \cap \alpha_j$ is a maximal initial segment of α_i such that $a \in \alpha_i \cap \alpha_j$.

Let $\alpha_i = [a,c]$. Let $\alpha_j = [a,d]$ with $\alpha_i \cap \alpha_j = [a,b]$ and note $b \in (a,c)$ since $[a,b]$ is a proper initial segment of $[a,c]$. Obtain a sequence $b_n \to b$ such that $b_n \in (b,c]$.

For each b_n obtain $k_n \neq i$ such that $b_n \in \alpha_{k_n}$. Since we have only finitely many closed arcs α_j, there exists k and N such that $[b, b_N] \subset \alpha_k$.

Recall $\alpha_k \cap \alpha_i$ is a proper initial segment of α_i and note $a \notin \alpha_k \cap \alpha_i$ since otherwise $l[a, b_N] > l([a,b])$ contradicting our choice of j.

Thus $[b,c] \subset \alpha_k$ and recall $k \neq i$.

Let $[x,c] = \alpha_k \cap \alpha_i$ with $x \in [a,b]$. Note $x \neq a$ since otherwise $\alpha_j \cap \alpha_i = \alpha_i$, contradicting the fact that $\alpha_i \cap \alpha_k$ is a proper subset of α_i. Since $a \in \alpha_j \backslash \alpha_k$, we know $[d,b] \subset \alpha_k$ and hence

$\alpha_j \cap \alpha_k = [x,d]$. Thus $x = b$ since otherwise α_k is not a topological arc (since $\alpha_k = [x,b] \cup [b,d] \cup [b,c]$). Thus $\alpha_i \cup \alpha_j \cup \alpha_k$ is a triod, contradicting Lemma 2. □

For each i, select a nonempty open arc $(w_i, v_i) \subset int(\alpha_i)$ such that $[w_i, v_i] \cap \alpha_j = \emptyset$ if $i \neq j$ and such that $\alpha_i \backslash (w_i, z_i)$ is the union of two arcs.

Let $\beta_1, \beta_2, ...\ \beta_{2(n-1)}$ denote the distinct arcs determined by $\alpha_i \backslash (w_i, z_i)$. Notice if $\beta_i \cap \beta_j \neq \emptyset$, then β_i and β_j share a common endpoint on $\cup(D_i)$ (since $\alpha_i \cap \alpha_j$ is does not disconnect α_i).

Moreover, since $[w_i, v_i] \cap \alpha_j = \emptyset$ β_j is not a subset of β_i. Consequently the arcs $\beta_1, ...\beta_{2n}$ are canonically partitioned under the equivalence relation $\beta_i\tilde{}\beta_j$ if $\partial\beta_i \cap \partial\beta_j \neq \emptyset$.

Rename the arcs β_i in format β_k^i such that $\{e_k\} \in (\cup D_i) \cap (\cap \beta_k^i)$. Thus e_k is the common endpoint in a given equivalence class.

By Lemma 5, for each k, the union of the arcs $\cup \beta_k^i$ is a tree.

Let $h : P \to P$ be a tiny homeomorphism fixing $\cup D_i$ pointwise and mapping $P \backslash (\cup D_i)$ into $int(P)$.

Notice $int(h(\beta_k^i)) \subset int(P)$. Consequently, (since $h(\beta_i) \cap h(\beta_i)) = \emptyset$ or $h(\beta_i) \cap h(\beta_i))$ is a final segment of each arc) in similar fashion to the proof of Theorem 3, it is apparent we can perturb, arbitrarily nearby, the interiors $int(\beta_k^i)$ while keeping fixed the endpoints $\partial\beta_k^i$, creating new arcs β_k^{*i} such that $int(\beta_k^{*i}) = int(\beta_l^{*j}) = \emptyset$ unless $k = l$ and $i = j$. We can also require that $int(\beta_k^{*j}) \cap (v_i, w_i) = \emptyset$ for all k, j, i.

Now, to obtain perturbations of the interiors of the original arcs $\{\alpha_i\}$, we assemble a perturbation of α_i as the union of the 3 arcs, $[w_i, v_i]$ and the corresponding arcs β_k^{*j} such that $w_i \in \beta_k^{*j}$, and β_l^{*t} such that $v_i \in \beta_k^{*j}$ and we let $\alpha_i^* = \beta_k^{*j} \cup [w_i, v_i] \cup \beta_l^{*t}$.

In this fashion we can perturb the arcs $\{\alpha_i\}$ and obtain arcs $\alpha_1^*,,..\alpha_{n-1}^*$ such that $int(\alpha_i^*) \cap int(\alpha_j^*) = \emptyset$, and $l(\alpha_i^*) < 2\varepsilon$ and such that $\partial\alpha_i = \partial\alpha_i^*$.

7.2.2. *Perturbing the endpoints of* $\{\alpha_i^*\}$. Recall we have arcs $\alpha_1^*,,..\alpha_{n-1}^*$ such that $int(\alpha_i^*) \cap int(\alpha_j^*) = \emptyset$, and $l(\alpha_i^*) < 2\varepsilon$ and such that $\partial\alpha_i^* \subset (\cup D_i)$.

To perturb the arcs $\{\alpha_i^*\}$ to be disjoint, let $\{e_1, .., e_m\} = \cup\partial\alpha_i^*$.

Thus $e_k \in D_i \backslash int(D_i)$.

For each e_k we may select a small open set $U_k \subset P$ such that $e_k \in U_k$ and such that $(\cup(\alpha_i^*)) \cap \overline{U}$ is a tree consisting of finitely many arcs $\beta_k^{*1}, ...\beta_k^{*n_k}$ intersecting such that $\beta_k^{*i} \cap \beta_k^{*j} = e_k$.

Let $\partial\beta_k^{*i} = \{e_k^i, b_k^i\}$ such that $e_k^i = e_k$ for all i.

It is topologically apparent the arcs $\{\beta_k^{*i}\}$ admit arbitrarily small perturbations in $\overline{U}$ (fixing b_k^i) into pairwise disjoint arcs β_k^{**i} such that $int(\beta_k^{**i}) \subset int(P)$ and such that $e_k^{**i} \in \cup(D_i)$.

As before, we assemble a perturbation α_i^{**} of α_i^* as the union of 3 arcs, a large closed subarc of $int(\alpha_i^*)$ and the two perturbations of the ends $\beta_k^{**i} \cup \beta_l^{**j}$.

Now we have pairwise disjoint arcs α_i^{**} such that $l(\alpha_i) < 2\varepsilon$. Moreover $\cup(D_i) \cup \alpha_1^{**} ... \cup \alpha_{n-1}^{**}$ is a nonseparating planar continuum as seen in the following Lemma.

Lemma 8. *Recall the arcs* $\alpha_1,,,\alpha_{n-1}$ *selected by our algorithm in the PL disk* P *such that* α_i^{**} *connects components of* $\cup D_i$. *Suppose we replace each arc* α_i *by an arc* α_i^{**} *such that* $\alpha_i^{**} \cap \alpha_j^{**} = \emptyset$ *and such that* α_i *and* α_i^{**} *connect the same components of* $\cup D_i$. *Then* $\cup D_i \cup \alpha_1^{**} .. \cup \alpha_{n-1}^{**}$ *is a nonseparating planar continuum.*

Proof. This follows by induction. By definition each component of $\cup D_i$ is cellular. Suppose each component of $\cup D_i \cup \alpha_1^{**} .. \cup \alpha_{k-1}^{**}$ is cellular. By construction the arc α_k^{**} connects distinct components cellular components D and E of $\cup D_i \cup \alpha_1^{**} ... \cup \alpha_{k-1}^{**}$ such that $\alpha_k^{**} \cap (D \cup E \cup \alpha_1^{**} .. \cup \alpha_{k-1}^{**}) = \partial\alpha_k^{**}$. It is topologically apparent that $D \cup \alpha_k^{**} \cup E$ is cellular. ☐

7.2.3. *Pushing* $\{\alpha_i^{**}\}$ *off of* $\{\gamma_j\}$. We assume D_i is the union of a PL disk E_i and finitely many pairwise disjoint arcs $\gamma_i^1, ...\gamma_i^{n_i}$ such that $E_i \cap \gamma_i^j = z_{ij} \in \partial\gamma_i^j$.

Suppose $\delta > 0$ such that $l(\gamma_i^j) < \delta$. We assume that $E_i \subset int(P)$ and that $int(\gamma_i^j) \subset int(P)$ and $\partial\gamma_i^j \backslash \{z_{ij}\} \subset \partial P$.

Recall we have pairwise disjoint arcs α_i^{**} such that $l(\alpha_i) < 2\varepsilon$ and $\cup(D_i) \cup \alpha_1^{**} ... \cup \alpha_{n-1}^{**}$ is a nonseparating planar continuum.

Unfortunately it is possible that $\partial\alpha_k^{**} \cap \gamma_i^j \neq \emptyset$ and we ultimately require that the intersection is empty.

Thus the task at hand is to slide the arcs α_k^{**} off of the arcs $\{\gamma_i^j\}$.

Unlike the previous perturbations (α_i^* and α_i^{**} could be chosen arbitrarily close to α_i) each end of α_i^{**} might have to move by an amount δ.

To see how to move the arcs α_i^{**}, first notice if $\alpha_i^{**} \cap \gamma_j \neq \emptyset$, then $\{z_i\} = \alpha_i^{**} \cap \gamma_j$ where $z_i \in \partial\alpha_i^{**}$.

Since each D_i is the union of a disk E_i with arcs γ_i^j attached to ∂E_i, for each γ_i^j we can select pairwise disjoint sets U_i^j (open in P) such that $\gamma_i^j \subset U_i^j$, such that $\overline{U_i^j} \cap (\gamma_i^j \cup E_i)$ is a topological triod (and in particular $\overline{U_i^j} \cap E_i \subset \partial E_i$), and such that if $\alpha_k^{**} \cap U_i^j \neq \emptyset$, then $\alpha_k^{**} \cap U_i^j$ is connected.

We can also demand that U_i^j is a PL topological disk, and that for each $x \in \overline{U}_i^j \backslash \gamma_i^j$ there exists a path in $\overline{U_i^j} \backslash \gamma_i^j$ of length less than δ connecting x to $\partial E_i \backslash \gamma_i^j$.

By construction , if $\alpha_k^{**} \cap \overline{U_i^j} \neq \emptyset$ then $\alpha_k^{**} \cap \overline{U_i^j}$ is an arc with precisely one end point $a_{kij} \in \gamma_i^j$ and the other endpoint $b_{kij} \in \partial U_i^j \backslash (\cup D_i)$. Hence, working entirely within $\overline{U_i^j}$, we can replace $\alpha_k^{**} \cap \overline{U_i^j}$ with an arc $\beta_{ki}^j \subset \overline{U_i^j}$ such that $l(\beta_{ki}^j) < \delta$ and such that $b_{kij} \in \partial\beta_{ki}^j$ and such that the other endpoint of β_{ki}^j is on ∂E_i and such that $int(\beta_{ki}^j) \cap (\cup D_i) = \emptyset$. It is apparent we can preserve the disjointness property.

Ultimately this procedure replaces the arcs α_k^{**} by arcs α_k^{***}, the union of 3 segments $\alpha_k^{***} = (\alpha_k^{**} \backslash (\beta_{ki}^j \cup \beta_{kl}^t)) \cup \beta_{ki}^j \cup \beta_{kl}^t$.

By construction $l(\alpha_k^{***}) < 2\varepsilon + 2\delta$, and $\partial\alpha_k^{***} \subset \cup E_i$, and $int(\alpha_k^{***}) \subset P \backslash (\cup D_i)$, and $\alpha_i^{***} \cap \alpha_j^{***} = \emptyset$ if $i \neq j$, and $(\cup E_i) \cup \alpha_1^{***} \cup .. \cup \alpha_{k-1}^{***}$ is a cellular planar continuum (since $\partial\alpha_i^{**}$ and $\partial\alpha_i^{***}$ connect the same components of $\cup D_i$).

8. Ingredients for proof of Theorem 10

8.1. **Standard planar Peano continua.** We clarify the structure of planar Peano continua and observe that any planar Peano continuum Y is homotopy equivalent to a 'thicker' planar Peano continuum X so that the components of $R^2 \backslash X$ have simple closed curve boundaries. However, the closure of this null sequence of circles (in some sense the 'boundary' of X) need not have locally path connected components. Similar observations are made in Theorem 2.4.1 [4].

Remark 2. *If $X \subset R^2$ then X is a Peano continuum if and only if the components $\{U_n\}$ of $(R^2 \cup \{\infty\}) \backslash X$ form a null sequence of simply connected open sets with locally path connected boundary. (On the one hand if X is a Peano continuum then each of the components of $R^2 \backslash X$ is open (since X is compact) and simply connected (since otherwise X fails to be connected) and $\{U_n\}$ is a sequence (since R^2 is separable and open sets have at most countably many components) and $diam(U_n) \to 0$ (since otherwise we can select a subsequence of large subcontinua $Z_n \subset U_n$, converging in the Hausdorff metric to a large subcontinuum $Z \subset X$ and there exists $z \in Z$ such that local connectivity of X fails) and ∂U_n is locally path connected (since $R^2 \backslash U_n$ is locally path connected). Conversely if X enjoys all of the above properties then X is compact (since $\cup U_n$ is open), and connected (since each U_n is open and simply connected), and locally path connected (since for large n, there is a small retract from $\overline{U_n \backslash u_n}$ onto ∂U_n, and if $\{x, y\} \subset X$, the short segment $[x, y]$ can be modified (replacing components of $[x, y] \cap U_n$ with the image in ∂U_n under the retraction) creating a small path in X from x to y).*

Recall if $X \subset R^2$ is a 2 dimensional Peano continuum then $int(X)$ is the maximal set $U \subset X$ such that U is open in R^2 and recall the frontier $Fr(X) = X \backslash int(X)$.

It is tempting to conclude that each component of $Fr(X)$ is locally path connected. However this is generally false as seen in the following example (see also [4] [12]).

Example 1. *Let Z be any 1 dimensional nonseparating planar continuum such that Z is not locally path connected. Manufacture a null sequence of pairwise disjoint simple closed curves $C_n \subset R^2 \backslash Z$ such that $\overline{\cup C_n} = Z \cup (\cup C_n)$ (and such that each C_n is an isolated subspace of the compactum $Z \cup (\cup C_n)$). Let U_n denote the bounded component of $R^2 \backslash C_n$ and define $X = R^2 \backslash (\cup U_n)$. Then X is a Peano continuum (since the components of $(R^2 \cup \{\infty\}) \backslash X$ form a null sequence of simply connected open sets with locally connected boundary) however Z is a component of $Fr(X)$ and Z is not locally path connected.*

It will prove useful to obtain a canonical form (up to homotopy equivalence) for planar Peano continua.

Definition 2. *Suppose $X \subset R^2$ is a 2 dimensional Peano continuum. Then X is **standard** if for each component $U \subset R^2 \backslash X$, ∂U is a round Euclidean circle such that ∂U is isolated in $Fr(X)$.*

Lemma 9. *Suppose $Y \subset R^2$ is a Peano continuum. Then there exists a standard Peano continuum X such that $Y \subset X$ and Y is a strong deformation retract of X. In particular Y is homotopy equivalent to X.*

Proof. Since R^2 is separable, the open subspace $R^2 \backslash X$ has at most countably many components. Moreover each component $U \subset R^2 \backslash X$ is simply connected, since X is connected. Let $\{U_n\}$ denote the simply connected components of $R^2 \backslash Y$. Note, for each simply connected U_n, ∂U_n is locally path connected. Select $u_n \in U_n$ and note ∂U_n is a strong deformation retract of $U_n \backslash \{u_n\}$. In particular we can select a simple closed curve $C_n \subset U_n$ approximating ∂U_n (let C_n denote the image of a large round circle $S_n \subset int(D^2)$ under a Riemann map $\overline{\phi} : \overline{int(D^2)} \to \overline{U_n}$). Let $A_n \subset \overline{U_n}$ denote the closure of the open annulus bounded by C_n and ∂U_n. Note we have a strong deformation from A_n onto ∂U_n with small trajectories under the homotopy. Let $Y = X \cup (\cup A_n)$. Since X is a Peano continuum, $diam(U_n) \to 0$. The space Y is a Peano continuum by Remark 2. Moreover, since $diam(A_n) \to 0$, the union of the deformation retracts from $A_n \to \partial U_n$ determines that X is a strong deformation retract of Y. By construction Y is standard. □

Lemma 10. *Suppose $X \subset R^2$ is a standard Peano continuum and $V \subset R^2 \backslash X$ is the union of some bounded components of $R^2 \backslash X$. Then $X \cup V$ is a standard Peano continuum.*

Proof. Let $Y = X \cup V$. Then if $u \in R^2 \backslash Y$ then there exists a component $U \subset R^2 \backslash X$ such that $u \in U$. Thus Y is compact and it follows from Remark 2 that Y is a Peano continuum whose complementary domain boundaries form a null sequence of circles $S_1, S_2, ...$

Recall S_n is isolated in $X \backslash int(X)$. To see that S_n is isolated in $Y \backslash int(Y)$, note $V \subset int(Y)$ and thus $Y \backslash int(Y) \subset X \backslash int(X)$. □

8.2. **Building and recognizing homotopic maps.** Notice if α and β are two unbased inessential loops in $R^2 \backslash \{(0,0)\}$ such that $diam(\alpha) < \varepsilon$ and $diam(\beta) < \varepsilon$ then $im(\alpha) \cup im(\beta)$ might have large diameter. However if α and β are homotopic

essential loops then we are guaranteed a small homotopy between α and β since both loops must stay near the 'hole' at $(0,0)$. (See also Theorem 2.1 [4])

The foregoing example illustrates a more general phenomenon captured by the following 2 Lemmas.

Our proof of Lemma 11 includes both a direct argument and a backhanded proof exploiting the nontrivial fact that the fundamental group of a planar Peano continuum Z injects into the inverse limit of free groups (determined by Z as the nested intersection of a sequence open planar sets).

Lemma 11. *Suppose $Y \subset R^2$ is any set and $\alpha, \beta : S^1 \to Y$ are essential homotopic loops such that $diam(im(\alpha)) < \varepsilon$ and $diam(im\beta) < \varepsilon$. Then $diam(im\alpha \cup im\beta) < 2\varepsilon$.*

Proof. Note $im(\alpha)$ is a Peano continuum. Let Z_α denote the union of $im(\alpha)$ and those components $U \subset R^2 \backslash im(\alpha)$ such that $U \cap Y = U$. Then $Z_\alpha \subset Y$ and Z_α is a Peano continuum (by Lemma 10). Since $Z_\alpha \subset Y$ it follows that α is essential in Z_α. Note $diam(Z_\alpha) < \varepsilon$. For each bounded component $V \subset R^2 \backslash Z_\alpha$ select a point $z_V \in V \backslash Y$ to obtain a set $E \subset R^2 \backslash (Z_\alpha \cup Y)$. Note $Y \subset R^2 \backslash E$. Thus α and β are homotopic in $R^2 \backslash E$.

To see that α is essential in $R^2 \backslash E$ it suffices to prove that Z_α is a strong deformation retract of $R^2 \backslash E$. (For each bounded component $V \subset R^2 \backslash Z_\alpha$, notice $\overline{V \backslash z_v}$ can be deformation retracted onto ∂V, and since the components of $R^2 \backslash Z_\alpha$ determine a null sequence it follows, taking the union of the $SDRs$, that Z_α is a strong deformation retract of $R^2 \backslash E$).

(Alternately we can obtain a finite set $E_n \subset E$ such that α is essential in $R^2 \backslash E_n$ as follows. Obtain PL nested compact polyhedra $..P_3 \subset P_2 \subset P_1$ such that $Z_\alpha \subset \cap_{n=1}^{\infty} P_n$. Inclusion determines a canonical homomorphism $\phi : \pi_1(Z_\alpha) \to \lim_{\leftarrow} \pi_1(P_n)$, it is a nontrivial fact that ϕ is injective (established more generally for planar sets [8]).Thus there exists N such that α is essential in P_N and such that $diam(P_N) < \varepsilon$, and now select a point from each bounded component of $R^2 \backslash P_N$ to obtain E_N).

Recall α and β are essential homotopic loops in $R^2 \backslash E$ and $diam(E) < \varepsilon$. Let $B \subset R^2$ denote the convex hull of the set E. Note $diam(B) = diam(E)$. It is apparent that $B \cap im(\beta) \neq \emptyset$, (since otherwise β would be inessential in $R^2 \backslash E$). Thus $diam(im\alpha \cup im\beta) < 2\varepsilon$. □

Lemma 12. *Suppose Y is a planar set, $\varepsilon > 0$, and A denotes the closed annulus $S^1 \times [0,1]$. Suppose $h : A \to Y$ is a map such that $diam(h(\partial A)) < \varepsilon$. Then there exists a map $H : A \to Y$ such that $diam(H(A)) < \varepsilon$ and $h_{\partial A} = H_{\partial A}$. Suppose $\alpha : S^1 \to Y$ is inessential and suppose $diam(im(\alpha)) < \varepsilon$. Then there exists a map $\beta : D^2 \to Y$ such that $\beta_{S^1} = \alpha$ and $diam(im(\beta)) < \varepsilon$.*

Proof. Let U be the unbounded component of $R^2 \backslash h(\partial A)$. Note $diam(P) < \varepsilon$ and P is a simply connected Peano continuum and hence there exists a retract $R : R^2 \to P$ such that $R_P = id_P$. Let $H = R(h)$.

Let V be the unbounded component of $R^2 \backslash im(\alpha)$. Let $Q = R^2 \backslash V$. Then Q is a simply connected Peano continuum and hence there exists a retract $r : R^2 \to Q$. Let $\gamma : D^2 \to Y$ be any map extending α and let $\beta = r(\gamma)$. □

The following elementary Lemma is essentially the Alexander Trick and ensures we can canonically adjust a map of a disk while keeping half the boundary unadjusted.

Lemma 13. *Suppose Y is any metric space and D is a topological disk and $p \in \partial D$ and $\gamma \subset \partial D$ is closed arc such that $p \notin \gamma$. Suppose $f : D \to Y$ is a map. Suppose $\alpha : [0,1] \to Y$ is a path connecting $f(p)$ and $q \in Y$. Then there exists a map $g : D \to Y$ such that $g(p) = q$ and $g_\gamma = f_\gamma$ and there exists a homotopy from f to g such that the trajectories in Y under the homotopy have diameter bounded by $diam(im(\alpha) \cup im(f))$. Moreover $diam(g(D)) \leq diam(f(D) \cup im(\alpha))$.*

Proof. We may assume $D \subset R^2$ is the closed upper half disk of radius 1 centered at $(0,0) = p$, and $\gamma \subset \partial D$ is the semicircle of radius 1.

Let $E \subset R^2$ denote the closed unit disk. Define $F : E \to Y$ so that $F(x,-y) = f(x,y)$. Notice for each $z_\theta \in \partial E$ there is a canonical path in Y connecting $f(z_\theta)$ and q, (we let β_θ denote the radial segment connecting z_θ and $(0,0)$ and we let $\gamma_\theta = f(\beta_\theta) * \alpha$.)

Define $G : E \to Y$ so that G maps $\beta_\theta \subset E$ linearly onto $\gamma_\theta \subset Y$. Let α_s denote a homotopy in Y from p to α so that $im(\alpha_s) \subset im(\alpha)$.

For $s \in [0,1]$ define $\beta^1_\theta(s)$ and $\beta^2_\theta(s)$ so that $\beta_\theta = \beta^1_\theta(s) * \beta^2_\theta(s)$ (concatenated segments varying linearly with s) so that $\beta^1_\theta(0) = \beta_\theta$ and $\beta^2_\theta(0) = (0,0)$. To obtain a homotopy from F to G let F_s map $\beta^1_\theta(s) * \beta^2_\theta(s)$ onto $f(\beta_\theta) * \alpha_s$ 'homomorphically'. Let $g = G_D$ and let $f_s = F_{sD}$. □

Theorem 6. *Suppose $f : X \to Y$ is a map of a standard Peano continuum X and suppose (Y,d) is any metric space. Suppose $\{x_n\}$ is a sequence of distinct points in X (and each x_n belongs to some isolated boundary circle $C_m \subset Fr(X)$ and each boundary circle contains at most finitely many of $\{x_n\}$) and suppose y_n is a sequence in $im(f) \subset Y$ and suppose $d(f(x_n), y_n) \to 0$. Then there exists a map $f\hat{} : X \to Y$ such that $f\hat{}$ is homotopic to f and $f\hat{}(x_n) = y_n$.*

Proof. Since $im(f)$ is a Peano continuum, there exists a null sequence of paths $\alpha_n : [0,1] \to im(f)$ connecting $f(x_n)$ and y_n. Select a null sequence of pairwise disjoint closed topological disks $\{D_n\}$ such that $x_n \in D_n \subset X$ and such that $(\partial D_n) \cap Fr(X)$ is a nontrivial arc containing x_n in its interior. Let $\gamma_n = \partial D_n \backslash (int((\partial D_n) \cap Fr(X)))$. Let $f\hat{}$ and f agree over the set $X \backslash (\cup int(D_n))$. Apply Lemma 13 to the data $(D_n, \gamma_n, x_n, \alpha_n)$, and sew together the resulting maps to obtain $f\hat{}$. □

Lemma 14. *Suppose $f, g : X \to Y$ are maps of a standard planar Peano continuum X into the metric space Y. Suppose for each isolated circle $C_n \subset Fr(X)$ there exists $x_n \in C_n$ such that $f(x_n) = g(x_n)$. Suppose $f_* = g_*$ and $f_*, g_* : \pi_1(X, x_1) \to \pi_1(Y, f(x_1))$ are the induced homomorphisms. Suppose $f(C_n)$ is essential in Y for each isolated circle $C_n \subset Fr(X)$. Then there exists a map $f\hat{} : X \to Y$ such that f is homotopic to f and $\hat{f}_{Fr(X)} = g_{Fr(X)}$.*

Proof. Since $f_* = g_*$, for each C_n the loops f_{C_n} and g_{C_n} are essential in Y and path homotopic in Y. It follows from Lemma 11 that $\lim diam_{n\to\infty}((f(C_n)) \cup g(C_n)) \to 0$.

It follows from Lemma 12 that there exists a path homotopy connecting $f(C_n)$ and $g(C_n)$ so that the image of the path homotopy has diameter bounded by $diam((f(C_n)) \cup g(C_n)) + \frac{1}{2^n}$.

Since X is a standard Peano continuum, we may select pairwise disjoint closed round disks $D_n \subset R^2$ such that $C_n \subset D_n$, such that $C_n \cap \partial D_n = \{x_n\}$ and such that $diam(D_n) \to 0$.

Note the based loops $f_{\partial D_n}, f_{C_n}$ and g_{C_n} are homotopic (fixing x_n throughout), and By Lemma 12 for large n the homotopies can be chosen to be small.

In particular, in the pinched annulus A_n bounded by $C_n \cup \partial D_n$ we can define $f\hat{}:A_n \to Y$ so that $f\hat{}_{\partial D_n} = f_{\partial D_n}$, and so that $f\hat{}_{C_n} = g_{C_n}$ and so that $diam(f\hat{}(A_n)) \to 0$. Let f and $f\hat{}$ agree on the set $X\backslash(\cup int(A_n))$ and we have obtained the desired map $f\hat{}$. □

Lemma 15. *Suppose $X \subset R^2$ is a standard Peano continuum, suppose Y is any metric space and $f : X \to Y$ a map. Suppose $U \subset R^2\backslash X$ is the union of those bounded components $\{U_n\} \subset R^2\backslash X$ such that $f_{\partial U_n}$ is inessential in Y. Let $Z = X \cup U$. Then Z is a standard Peano continuum and there exists a map $F : Z \to Y$ such that $F_X = f$ and F_{C_n} is essential for all map isolated circles $C_n \subset Fr(Z)$.*

Proof. By Lemma 10 that Z is a standard Peano continuum.

For each component $U_n \subset U$, the unbased loop $f_{\partial U_n}$ inessential in Y, and thus there exists an extension $F_{\overline{U_n}} \to Y$ such that $F_{\partial U_n} = f_{\partial U_n}$. By Lemma 12 we can also require that $diam(F(U_n)) \leq diam(f(\partial U_n))$. Let F and f agree over X, since $diam(U_n) \to 0$ and since f is uniformly continuous on X, it follows that $diam(f(U_n)) \to 0$ and hence the extension F is continuous. By construction $f_{C_n} = F_{C_n}$ for all isolated circles $C_n \subset \partial Fr$ and hence F_{C_n} is essential in Y. □

Lemma 16. *Suppose $X \subset R^2$ is a standard Peano continuum, suppose Y is any metric space and $p \in X$ and $f, g : X \to Y$ are maps such that $f(p) = g(p)$ and $f_* = g_*$ and $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ is the induced homomorphism. Suppose $U \subset R^2\backslash X$ is the union of those bounded components $\{U_n\} \subset R^2\backslash X$ such that $f_{\partial U_n}$ is inessential in Y. Suppose $Z = X \cup U$ and suppose the maps $F, G : Z \to Y$ satisfy $F_X = f$ and $G_X = g$. Then $G_* = F_*$ (and $F_* : \pi_1(Z, p) \to \pi_1(Y, f(p))$ denotes the induced homomorphism).*

Proof. Suppose $\alpha : [0, 1] \to Z$ is a loop based at p. By Lemma 10 Z is a Peano continuum. For each component $U_n \subset U$ and each component $J \subset \alpha^{-1}(U_n)$ replace α_J by β_J such that $\beta_{\overline{J}}$ is path homotopic to $\alpha_{\overline{J}}$ in Z and such that $im(\beta_J) \subset \partial U_n$ and such that $diam(im(\alpha_j)) = diam(im(\beta_j))$. Let $\beta = \cup_J \alpha_{([0,1]\backslash \cup J)} \cup \beta_J)$.

Since the collection of all such open arcs J is a null sequence of intervals in $[0, 1]$, and since α is uniformly continuous, the homotopies connecting α_J to β_J can be chosen to be small, and the union of the homotopies determines that α and β are path homotopic in Z. Thus $F(\alpha)$ and $F(\beta)$ are path homotopic in Y and $G(\alpha)$ and $G(\beta)$ are path homotopic in Y.

Note $f(\beta) = F(\beta)$ and $g(\beta) = G(\beta)$ and, (since $f_* = g_*$), $f(\beta)$ and $g(\beta)$ are path homotopic in Y. Thus $F(\alpha)$ and $G(\alpha)$ are path homotopic in Y. □

Lemma 17. *Suppose X is a planar continuum and $Y \subset R^2$ is any planar set and $Z \subset X$ is a continuum and $f, g : X \to Y$ are maps such that $f_Z = g_Z$. Then f and g are homotopic if both the following conditions hold 1) Each bounded component $U \subset X\backslash Z$ is an open Jordan disk. (i.e. $\overline{U}\backslash U$ is a simple closed curve) and 2) If $X\backslash Z$ has infinitely many components $U_1, U_2, ..$ then $diam(U_n) \to 0$.*

Proof. For each bounded component $U \subset X\backslash Z$ select embeddings $h_u : \overline{U} \hookrightarrow \partial B^3$ and $h_l : \overline{U} \to \partial B^3$ mapping $\overline{U}$ respectively onto the upper and lower hemispheres of the 2$-$sphere. Glue the maps together to obtain a map of the 2-sphere $j_U = f(h_u^{-1}) \cup g(h_l^{-1}) : \partial B^3 \to Y$. Since planar set are aspherical [3], j_U is the restriction of a map $J_U : B^3 \to im(j_U)$. The map J_U determines a homotopy ϕ^t_U between $f_{\overline{U}}$ and $g_{\overline{U}}$ and the image of the homotopy lies in $im(f_{\overline{U}}) \cup im(g_{\overline{U}})$. If $X\backslash Z$ has

finitely many components then we take the union of the homotopies. Since each of f and g are uniformly continuous (since X is compact) , and since $diam(U_n) \to 0$, it follows that the union of the homotopies $f_z \cup (\cup_n \phi^t_{U_n})$ determines a global homotopy between f and g. □

Lemma 18. *Suppose $X \subset R^2$ is a 2 dimensional continuum and $\beta_1, \beta_2, ...$is null sequence of closed arcs such that $int(\beta_n) \subset int(X)$, and $\partial\beta_n \subset Fr(X)$ and such that $int(\beta_n) \cap int(\beta_m) = \emptyset$ if $n \neq m$. Suppose Y is a planar set. Suppose $f, g : X \to Y$ are maps such that $f_{Fr(X)} = g_{Fr(X)}$. Suppose $p \notin \cup\beta_n$ and $f(p) = g(p)$. Suppose $f_* = g_*$ and $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ is the induced homomorphism. Let $Z = Fr(X) \cup (\cup\beta_n)$. Then there exists a map $f\hat{}: X \to Y$ such that $f\hat{}$ is homotopic to f and $f\hat{}_Z = g_Z$.*

Proof. 'Thicken' each β_n into a closed topological disk D_n (such that $p \notin D_n$) and $int(\beta_n) \subset int(D_n) \subset int(X)$, and $D_n \cap Fr(X) = \partial\beta_n$, and if $n \neq m$ then $D_n \cap D_m \subset \partial\beta_n \cup \partial\beta_m$, and $diam(D_n) \to 0$. Since $f_* = g_*$ it follows that f_{β_n} and g_{β_n} are path homotopic in Y (fixing $\partial\beta_n$ throughout the homotopy).

Note β_n is a strong deformation retract of D_n and if (the topological semicircle) γ is the closure of either component of $(\partial D_n)\backslash(\partial\beta_n)$ then f_γ is path homotopic to f_{β_n}. Consequently we can define $f\hat{}_{D_n} : D_n \to Y$ so that $f\hat{}_{\partial D_n} = f_{\partial D_n}$ and so that $f\hat{}_{\beta_n} = g_{\beta_n}$. By Lemma 12 we can also arrange $diam(f\hat{}(D_n)) < diam(f(\partial D_n) \cup g(\beta_n))$.

Now sew together two copies of D_n joined along ∂D_n and notice $f\hat{}_{D_n} \cup f_{D_n} : S^2 \to Y$ determines a map of the 2 sphere into Y. Moreover $Z_n = f\hat{}(D_n) \cup f(D_n)$ is a Peano continuum (since S^2 is a Peano continuum). Since Z_n is aspherical ([3]) $f\hat{}_{D_n}$ and f_{D_n} are homotopic in Z_n (via a homotopy f^t_n such that $f^t_{n|\partial D_n} = f_{\partial D_n}$).

Define $f\hat{}_{X\backslash(\cup int(D_n))} = f_{X\backslash(\cup int(D_n))}$. For large n the homotopy f^t_n has small image (since f and g are uniformly continuous and both D_n and Z_n are null sequences).

Thus the union of the homotopies f^t_n shows $f\hat{}$ and f are homotopic. □

8.3. **Chopping up simply connected sets into small disks with crosscuts.** Given a PL planar disk D with finitely marked points $Y \subset \partial D$ we wish to partition D using crosscuts (with disjoint interiors) connecting distinct points of Y, and to obtain control of the diameter of the regions bounded by the crosscuts (Theorem 7). Combined with a standard construction from the theory of prime ends (Lemma 22), we see how to subdivide simply connected open sets $U \subset S^2$ into a null sequence of topological disks whose boundaries contain points of ∂U (Theorem 8).

Define the planar set $Y \subset R^2$ as 'ε thin' if (letting $B(x, \varepsilon) \subset R^2$ denote the round open disk of radius ε) for each $x \in Y$, $B(x, \varepsilon) \cap (R^2\backslash Y) \neq \emptyset$.

If D is a topological disk an arc $\alpha \subset D$ is a **spanning arc** if $int(\alpha) \subset int(D)$ and $\partial\alpha \subset \partial D$. Let $S(v, \varepsilon)$ denote the round circle of radius ε centered at v.

Lemma 19. *Suppose $...D_3 \subset D_2 \subset D_1 \subset R^2$ is a sequence of closed topological disks. Suppose $\varepsilon > 0$. Then there exists N so that if $n \geq N$ and U is a component of $D_n\backslash D_{n+1}$ then U is ε thin.*

Proof. Suppose otherwise to obtain a contradiction. There exists an increasing sequence $n_k \to \infty$ and $u_{n_k} \in U_{n_k}$ so that U_{n_k} is a component of $D_{n_k}\backslash D_{n_k+1}$ and $B(u_{n_k}, \varepsilon/2) \subset U_{n_k}$. Let z be a subsequential limit of $\{u_{n_k}\}$. Then $B(z, \frac{\varepsilon}{4}) \subset U_{n_k}$ for all sufficiently large k. Thus $z \notin \cap^\infty_{n=1} D_n$ and $z \in \cap^\infty_{n=1} D_n$ and we have a contradiction. □

If $S \subset R^2$ is a simple closed curve and $P \subset S$ is finite then P is '$\varepsilon-$ dense' in S if for each pair of consecutive clockwise points $x < y \in P$ each clockwise arc $\alpha_{xy} \subset S$ connecting x to y satisfies $diam(\alpha_{xy}) < \varepsilon$.

The inequalities in Lemma 20 and Theorem 7 are not sharp.

Lemma 20. *If $D \subset R^2$ is any ε thin topological disk, then there exist finitely many pairwise disjoint spanning arcs $\alpha_1, \alpha_2, ..$ such that if U is a component of $D^2 \backslash (\cup \alpha_n)$, then $diam(U) < 12\varepsilon$.*

Proof. Tile the plane by squares of sidelength 5ε. Note there exist arbitrarily small perturbations E of D so that E is a PL disk, and so that if $v \in T$ is a corner of the tile T then $v \notin \partial E$, and so that each component $A \subset E \cap \overline{B(v, 2\varepsilon)}$ is a closed topological disk. Thus, wolog we may assume D also enjoys the aforementioned properties of E. Suppose v is the corner of a tile T and suppose $v \in int(D)$. Then, since D is ε thin, let the open arc γ be a nonempty component of $S(v, \varepsilon) \backslash D$ and let B denote the component of $\overline{B(v, \varepsilon)} \backslash D$ such that $\gamma \subset B$. Now manufacture a homeomorphism $h : \overline{B(v, \varepsilon)} \to \overline{B(v, \varepsilon)}$ (fixing $\partial \overline{B(v, \varepsilon)}$ pointwise) so that $v \in h(B)$. Applying this construction at the corner of each tile, ultimately we can obtain a 2ε homeomorphism $h : R^2 \to R^2$ so that $h(D)$ is a PL disk and so that $v \notin h(D)$ for all tile corners v, and so that $\partial h(D)$ crosses each tile edge finitely many times and transversely.

Let $E = h(D)$. Recall the tiles $T_1, T_2, ...$and note each component $A \subset E \backslash (\cup_{n=1}^{\infty} \partial T_n)$ satisfies $diam(A) < 5\sqrt{2}\varepsilon$. We obtain spanning arcs $\beta_1, \beta_2, , ,$ taking the components of $E \cap (\cup_{n=1}^{\infty} (\partial T_n))$. Now let $\alpha_n = h^{-1}(\beta_n)$. Let U be a component of $D \backslash (\cup \beta_n)$ and note $diam(h^{-1}U) < 5\sqrt{2}\varepsilon + 4\varepsilon < 12\varepsilon$. $\square$

Lemma 21. *Suppose the finite set P is ε dense in ∂D and $Q \subset \partial D$ is finite. Then there exists a monotone map:$f : D \to D$ such that $f(Q) \subset P$ and $d(f(x), x) < \varepsilon$ for all $x \in D$ and such that f maps $int(D)$ homeomorphically onto $int(D)$.*

Proof. For each $q \in Q$ let $f(q) = p \in P$ so that p is the nearest clockwise neighbor to q. For each $p \in P$ select an arc $\gamma_p \subset \partial D$ so that $p \cup f^{-1}(p) \subset int(\gamma_p)$ and so that $diam(\gamma_p) < \varepsilon$. We can also arrange that $\gamma_p \cap \gamma_r = \emptyset$ if $p \neq r$. Thicken the arcs $\{\gamma_p\}$ into a closed pairwise disjoint topological disks $\{D_p\}$ such that $\cup_{p \in P} D_p \subset D$. Let f fix $D \backslash (\cup D_p)$ pointwise. For each γ_p let $\beta_p \subset int(\gamma_p)$ be closed arc such that $p \cup f^{-1}(p) \subset int(\gamma_p)$. Let f fix ∂D_i pointwise, let f map β_p to p and let f map $D_p \backslash \beta_p$ homeomorphically onto $D_p \backslash p$. $\square$

Theorem 7. *There exists $M > 0$ so that if $D \subset R^2$ is an ε thin topological disk and the finite set $P \subset \partial D$ is ε dense then there exist spanning arcs $\beta_1, \beta_2, ...$ such that $int(\beta_n) \cap int(\beta_m) = \emptyset$ and such that $P = \cup(\partial \beta_n)$ and such that each component $U \subset D^2 \backslash (\cup \beta_n)$ is an open Jordan disk and $diam(U) < 14\varepsilon$.*

Proof. Obtain disjoint closed spanning arcs $\alpha_1, \alpha_2,$ as in Lemma 20 so that each component $U \subset D^2 \backslash (\cup \alpha_n)$ has diameter at most 12ε.

Apply Lemma 21 to obtain a monotone map:$f : D \to D$ such that $f(\cup \alpha_n) \subset P$ and $d(f(x), x) < \varepsilon$ for all $x \in D$ and such that f maps $int(D)$ homeomorphically onto $int(D)$.

Let $\beta_n = f(\alpha_n)$. If U is a component of $D \backslash (\cup \alpha_n)$ then $diam(f(U)) < 12\varepsilon + 2\varepsilon$. At this stage it is likely that $P \backslash (\cup \partial \beta_n) \neq \emptyset$. In this case we merely add more spanning arcs to the collection $\beta_1, \beta_2, ..$ connecting any remaining points of P. $\square$

Definition 3. *Suppose $U \subset S^2$ is open and simply connected. Recall a* ***closed crosscut*** *γ is a closed nontrivial topological arc such that $\gamma \subset \overline{U}$ and $int(\gamma) \subset U$ and $\partial\gamma \subset \partial U$. By a loop of* ***concatenated crosscuts*** *$\gamma_1, \gamma_2, ..\gamma_n$ we mean each γ_i is a closed crosscut of U and $int(\gamma_i) \cap int(\gamma_j) = \emptyset$ if $i \neq j$ and $\cup\gamma_i$ is a simple closed curve.*

Suppose $V \subset R^2$ is open, bounded, and simply connected and $\delta > 0$. Let $T_1^\delta.,T_2^\delta,...$ be a tiling of R^2 by squares of sidelength δ. Let $A_\delta \subset V$ be a maximal closed topological disk consisting of the union of finitely many closed tiles. For small δ it is apparent that each point of ∂A can be connected to ∂V within V by a crosscut of diameter at most 2δ (since otherwise we could attach another tile to A). Consequently we have the following Lemma which will be obvious to the reader familiar with prime ends.

Lemma 22. *Suppose $X \subset R^2$ is a nonseparating continuum and α is a crosscut of $R^2 \backslash X$ and V is the bounded component of $R^2 \backslash (X \cup \alpha)$. Suppose $\varepsilon > 0$. There exist finitely many crosscuts $\beta_1, \beta_2, ...$ of V such that $\alpha \cup (\cup \beta_i)$ is a simple closed curve and $diam(\beta_i) < \varepsilon$.*

Given a cellular continuum $X \subset R^2$ and applying Lemma 22 recursively (applied to $\varepsilon_n = \frac{1}{2^n}$) we can manufacture a sequence of closed topological disks $...D_3 \subset D_2 \subset D_1 \subset R^2$ such that 1) $X = \cap_{n=1}^\infty D_n$ and 2) each component $V \subset D_n \backslash D_{n+1}$ is an open planar set such that ∂V is a loop of finitely many crosscuts (of $R^2 \backslash X$) $\gamma_1, \gamma_2, ...$ such that $\gamma_1 \subset \partial D_n$, and $diam(\gamma_1) < \varepsilon_n$, and $\gamma_2 \cup \gamma_3.. \subset \partial D_{n+1}$, and $diam(\gamma_i) < \varepsilon_{n+1}$ if $i \geq 2$.

By Lemma 19, given the disks $D_1, D_2, ...$ and $\varepsilon > 0$ notice there exists N so that if $n \geq N$ then $D_n \backslash D_{n+1}$ has ε thin (components).

Moreover by construction for each component $V \subset D_n \backslash D_{n+1}$, ∂V is decorated by an ε_n dense finite subset (the endpoints of the crosscuts thus far selected) and hence we may apply Theorem 7 to V to obtain the following Theorem.

Theorem 8. *Suppose $X \subset R^2$ is a nonseparating continuum. Then there exists a null sequence of crosscuts $\{\beta_n\}$ such that $int(\beta_n) \subset R^2 \backslash X$ and $\partial\beta_n \subset X$ and such that $int(\beta_n) \cap int(\beta_m) = \emptyset$ if $m \neq n$ and such that the components $\{U_n\}$ of $R^2 \backslash (X \cup (\cup\beta_n))$ form a null sequence of open sets (with simple closed curve boundaries), and for each U_n, the simple closed curve ∂U_n is the finite union of concatenated arcs β_n.*

9. Main results

Theorem 9. *Suppose $X \subset R^2$ is compact and $U = R^2 \backslash X$ is connected. Suppose X has at least two components. Then there exists a sequence of arcs $\alpha_1, \alpha_2, ...$ such that $l(\alpha_i) \to 0$, and $Z = X \cup (\cup_{i=1}^\infty \alpha_i)$ is cellular, and $int(\alpha_i) \subset U$, and α_i connects distinct components of X, and $int(\alpha_i) \cap int(\alpha_j) = \emptyset$ if $i \neq j$.*

Proof. Let $\delta_n = \frac{1}{10^n}$.

Apply Theorem 1 to obtain a sequence of closed sets $S_n \subset R^2$ such that S_n is the union of finitely many pairwise disjoint closed PL topological disks, such that $S_{n+1} \subset int(S_n)$, such that $X = \cap_{n=1}^\infty S_n$ and such that $N(S_n, S_{n+1}) < \delta_n$ and such that $\lim_{n\to\infty} M(S_n, S_{n+1}) = 0$. We require that S_1 is a connected PL disk.

Name a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 2M(S_n, S_{n+1}) + 2\delta_n$ and such that $\varepsilon_n \to 0$.

Let $Y_1 = \emptyset$ and proceed recursively as follows.

Suppose $Y_n = \{y_n^1, ...y_n^{k_n}\} \subset \partial S_n$ is finite. Apply Theorem 3 to obtain pairwise disjoint PL closed arcs $\{\gamma_n^i\} \subset S_n$ such that $int(\gamma_n^i) \subset int(S_n) \backslash S_{n+1}$ and γ_n^i connects y_n^i to S_{n+1} and $l(\gamma_i) < \delta_n$.

Recall each component $P_i^n \subset S_n$ is a PL disk, and the subspace $S_{n+1} \cup_{i,j} \{\gamma_j^i\} \subset S_n$ is the union of pairwise disjoint PL cellular sets.

Now apply Theorem 4 to the data at hand as follows.

Apply the algorithm in section 7 to the data $(S_n, S_{n+1} \cup_{i,j} \{\gamma_j^i\})$ creating a finite sequence of arcs $\{\alpha_n^i\} \subset S_n$ with the following properties: $l(\alpha_n^i) \leq 2M(S_n, S_{n+1})$, and $\alpha_n^i \cap \alpha_n^j$ is connected, and if $i \neq j$ then $\alpha_n^i \cap \alpha_n^j$ does not disconnect α_n^i, and α_n^i connects distinct components of $S_{n+1} \cup_{i,j} \{\gamma_j^k\}$.

Next apply the constructions in section 7.2, replacing the arcs α_n^i with pairwise disjoint arcs $\beta_n^i \subset S_n$ (replacing the notation α_n^{***i}) with all of the following properties:

$l(\beta_n^i) < 2M(S_n, S_{n+1}) + 2\delta_n$, $\beta_n^i \cap \gamma_k^j = \emptyset$, and β_n^i connects the same two distinct components of $S_{n+1} \cup_{i,j} \{\gamma_j^k\}$ (as α_n^i), and $int(\beta_n^i) \subset int(S_n)$, and each component of $S_{n+1} \cup (\cup_i \beta_n^i)$ is cellular.

Now let $Y_{n+1} = \partial S_{n+1} \cap ((\cup_i \beta_n^i) \cup (\cup_i \gamma_n^i)$ and repeat the construction.

(It is allowed at a given stage n, that $\cup_i \beta_n^i = \emptyset$, (in the event that S_n and S_{n+1} have the same number of components, and in fact this behavior is inevitable if X has finitely many components)).

To understand the components of $Z \backslash X$, by construction at each stage n, new arcs $\{\beta_n^i\}$ are created such that $l(\beta_n^i) < \varepsilon_n$. In subsequent stages a given end of β_n^i will be lengthened by attaching concatenated arcs $\gamma_{n+1}^{(n,i)} \cup \gamma_{n+2}^{(n,i)} ...$ (and on the other end of β_n^i we have concatenated arcs $\gamma_{n+1}^{*(n,i)} \cup \gamma_{n+2}^{*(n,i)} ...$).

Thus, $\beta_n^i \cup \gamma_{n+1}^{(n,i)} \cup \gamma_{n+2}^{(n,i)} ... \cup (\gamma_{n+1}^{*(n,i)} \cup \gamma_{n+2}^{*(n,i)} ...)$ is an open arc with Euclidean pathlength less than $2\varepsilon_n + \Sigma_{k=n}^{\infty} \frac{2}{10^k} < 2\varepsilon_n + \frac{1}{2^n}$.

Define $\kappa_n^i = \overline{\beta_n^i \cup \gamma_{n+1}^{(n,i)} \cup \gamma_{n+2}^{(n,i)} ... \cup (\gamma_{n+1}^{*(n,i)} \cup \gamma_{n+2}^{*(n,i)} ...)}$. Note the ends of the open arc $\beta_n^i \cup \gamma_{n+1}^{(n,i)} \cup \gamma_{n+2}^{(n,i)} ... \cup (\gamma_{n+1}^{*(n,i)} \cup \gamma_{n+2}^{*(n,i)} ...)$ converge since this open arc has finite geometric length.

Moreover, since the extended ends of β_n^i will be forever trapped in distinct components of S_n, κ_n^i is a closed arc, (as a opposed to a simple closed curve).

Define $Z = X \cup (\cup_{n.i} \kappa_n^i)$.

Recall $\gamma_n^j \subset S_n$, and $M(S_n, S_{n+1}) \to 0$ and $X = \cap S_n$. Thus $\partial \kappa_n^i \subset X$.

By construction, Z is the nested intersection of the cellular sets $S_{n+1} \cup ((\cup_{k \leq n} \beta_k^i) \cup (\cup_{k \leq n} \gamma_k^i))$. Consequently Z is cellular.

By construction $int(\kappa_n^i) \cap int(\kappa_m^j) = \emptyset$ (if $n \neq m$ or $i \neq j$).

Reindex the arcs doubly indexed sequence $\{\kappa_n^i\}$ as $\alpha_1, \alpha_2, ...$ to obtain the desired arcs. □

If $X \subset S^2$ is compact then $S^2 \backslash X$ has at most countably many components $U_1, U_2, ..$, and we apply Theorem 9 to each component $U_n \subset S^2 \backslash X$ to obtain the following result applicable to all planar compacta.

Corollary 1. *Suppose $X \subset R^2$ is compact. Then there exists a sequence of closed arcs $\alpha_1, \alpha_2, ..$ such that $int(\alpha_i) \subset R^2 \backslash X$, and $\partial \alpha_i \subset X$ and $\lim_{i \to \infty} l(\alpha_i) = \emptyset$, and $int(\alpha_i) \cap int(\alpha_j) = \emptyset$ if $i \neq j$, and if $Y = X \cup (\cup \alpha_i))$ and if U is a component of $R^2 \cup \{\infty\} \backslash Y$ then U is simply connected, and each component of $R^2 \backslash X$ contains precisely one component of $R^2 \backslash Y$.*

Theorem 10. *Suppose $X \subset R^2$ is a Peano continuum and $Y \subset R^2$ is any set. Suppose $p \in X$ and $f, g : X \to Y$ are maps such $f(p) = g(p)$. Then f and g are homotopic if and only if $f_* = g_*$ (and $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ denotes the induced homomorphism between fundamental groups.)*

Proof. If $f\tilde{}g$ it is immediate that $f_* = g_*$. Conversely suppose $f_* = g_*$. Our goal is to prove that f and g are homotopic, and throughout the proof we will replace f by a homotopic map $f\hat{}$ with nicer properties, and for convenience we will then rename $f = f\hat{}$.

We reduce to the assumption that X is standard as follows.

Apply Lemma 9 to obtain a standard Peano continuum Z and a retraction $r : Z \to X$ such that r is homotopic to id_Z. Moreover $(fr)_* = (gf)_*$ and we hope to prove fr is homotopic to gr. If we find such a homotopy from Z then we can restrict to X to obtain a homotopy from f to g. Thus, renaming Z as X, we have reduced to the special case that X is a standard Peano continuum.

Suppose $X \subset R^2$ is a standard Peano continuum. Let $U \subset R^2 \backslash X$ denote the union of those bounded components $\{U_n\} \subset R^2 \backslash X$ such that $f_{\partial U_n}$ is inessential in Y. Let $Z = X \cup U$. By Lemma 15 Z is a standard Peano continuum and there exists a map $F : Z \to Y$ such that $F_X = f$ (in particular F_{C_n} is essential for all isolated circles $C_n \subset Fr(Z)$)

In similar fashion we can construct a map $G : Z \to Y$ such that $G_X = g$.

By Lemma 16 $F_* = G_*$ and if we can prove F is homotopic to G then, restricting the homotopy to X, we will have a homotopy from f to g.

Thus, once again renaming $Z = X$, we have reduced the problem to the further specialized assumption that $f(C)$ is essential in Y for all isolated boundary circles $C \subset Fr(X)$ (and $X \subset R^2$ is a standard Peano continuum).

Let $C_1, C_2 ...$ denote the isolated circles of $Fr(X)$ and for each n select a basepoint $x_n \in C_n$.

Let $y_n = g(x_n)$. By uniform continuity $diam(g(C_k)) \to 0$ and $diam(f(C_k)) \to 0$. Since $f_* = g_*$ we know the unbased loops f_{C_k} and g_{C_k} are essential and freely homotopic in Y.

Thus it follows from Lemma 11 that $d(x_n, y_n) \to 0$. Now apply Theorem 6 to obtain a map $f\hat{} : X \to Y$ such that $f\hat{}$ is homotopic to f and such that $f\hat{}(x_n) = g\hat{}(x_n)$.

Thus, wolog we may rename $f = f\hat{}$ and we assume henceforth that $f(x_n) = g(x_n)$.

Now apply Lemma 14 to obtain a map $f\hat{} : X \to Y$ such that $f\hat{}_{Fr(X)} = g_{Fr(X)}$ and such that f and $f\hat{}$ are homotopic.

Once again, we may rename $f\hat{} = f$ and henceforth assume $f_{Fr(X)} = g_{Fr(X)}$.

Apply Corollary 1 to obtain a null sequence of arcs $\beta_1, \beta_2, ...$ such that $int(\beta_n) \subset int(X)$ and $\partial\beta_n \subset Fr(X)$, and $int(\beta_n) \cap int(\beta_m) = \emptyset$ if $n \neq m$, and so that the components of $int(X) \backslash (\beta_1 \cup \beta_2 ...)$ are simply connected open planar sets and so that the endpoints of β_n belong to distinct components of $Fr(X)$.

Let $Z = Fr(X) \cup \beta_1 \cup \beta_2 .....$

For each bounded component $U_n \subset R^2 \backslash Z$, apply Theorem 8 to obtain a null sequence of crosscuts $\alpha_1^n, \alpha_2^n, ..$ so that $int(\alpha_k^n) \subset U_n$ and $\partial\alpha_k^n \subset \partial U_n$ and such that $int(\alpha_k^n) \cap int(\alpha_m^n) = \emptyset$ if $k \neq n$ and such that the components $\{V_n\}$ of $\overline{U_n} \backslash (\alpha_1^n, \alpha_2^n, ..)$ form a null sequence of open sets (with simple closed curve boundaries), and for each B_n, the simple closed curve ∂V_n is the finite union of concatenated arcs α_n^k.

Now let $\alpha_1, \alpha_2, ...$ denote the arcs $\cup_m\{\beta_m\} \cup_{n,k} (\alpha_k^n)$. By construction $\partial\alpha_n \subset Fr(X)$ and $int(\alpha_n) \cap int(\alpha_m) = \emptyset$ if $m \neq n$ and $\{\alpha_n\}$ is a null sequence of arcs. Let $Z = X \cup \alpha_1 \cup \alpha_2 ....$

Apply Lemma 18 to obtain a map $f\hat{} : X \to Y$ such that $f\hat{}$ is homotopic to f and $f\hat{}_Z = g_Z$. As before rename $f = f\hat{}$.

It follows now from Lemma 17 that f and g are homotopic. □

Consequently we obtain the following version of Whitehead's Theorem for planar Peano continua.

Corollary 2. *If $X, Y \subset R^2$ are Peano continua, then a map $f : X \to Y$ is a homotopy equivalence if there exists $p \in X$ and $q \in Y$ such that $f_* : \pi_1(X,p) \to \pi_1(Y,q)$ is an isomorphism and such that $f_*^{-1} : \pi_1(Y,q) \to \pi_1(X,p)$ is induced by a map.*

Proof. Let $g : (Y,q) \to (X,p)$ be such that $(fg)_* = f_*g_* = id_{\pi_1(Y,q)}$ and $(gf)_* = g_*f_* = id_{\pi_1(X,p)}$. Then by Theorem 10 gf and fg are homotopic to the respective identities. Hence f is a homotopy equivalence. □

References

[1] Blokh, Alexander; Misiurewicz, Michal; Oversteegen, Lex Planar finitely Suslinian compacta. Proc. Amer. Math. Soc. 135 (2007), no. 11, 3755–3764

[2] Bridson, Martin R.; Haefliger, André Metric spaces of non-positive curvature. (Grundlehren der Mathematischen Wissenschaften), 319. Springer-Verlag, Berlin, 1999.

[3] Cannon, J. W.; Conner, G. R.; Zastrow, Andreas One-dimensional sets and planar sets are aspherical. In memory of T. Benny Rushing. Topology Appl. 120 (2002), no. 1-2, 23–45.

[4] Cannon, J. W.; Conner, G. R. The homotopy dimension of codiscrete subsets of the 2-sphere S^2. Fund. Math. 197 (2007), 35–66.

[5] de Smit, Bart The fundamental group of the Hawaiian earring is not free. Internat. J. Algebra Comput. 2 (1992), no. 1, 33–37.

[6] Eda, K. The fundamental groups of one-dimensional spaces and spatial homomorphisms. Topology Appl. 123 (2002), no. 3, 479–505.

[7] Eda, K. Homotopy Types of 1 dimensional Peano continua. Preprint.

[8] Fischer, Hanspeter; Zastrow, Andreas The fundamental groups of subsets of closed surfaces inject into their first shape groups. Algebr. Geom. Topol. 5 (2005), 1655–1676

[9] Morgan, John W.; Morrison, Ian A van Kampen theorem for weak joins. Proc. London Math. Soc. (3) 53 (1986), no. 3, 562–576.

[10] Whitehead, J. H. C. Combinatorial homotopy. I. Bull. Amer. Math. Soc. 55, (1949). 213–245.

[11] Whitehead, J. H. C. Combinatorial homotopy. II. Bull. Amer. Math. Soc. 55, (1949). 453–496.

[12] Karimov, U.; Repov, D.; Rosicki, W.; Zastrow, A. On two-dimensional planar compacta not homotopically equivalent to any one-dimensional compactum. Topology Appl. 153 (2005), no. 2-3, 284–293.

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